Periodic solutions for a class of nonlinear partial differential equations in higher dimension

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Abstract

We prove the existence of periodic solutions in a class of nonlinear partial differential equations, including the nonlinear Schrödinger equation, the nonlinear wave equation, and the nonlinear beam equation, in higher dimension. Our result covers cases where the bifurcation equation is infinite-dimensional, such as the nonlinear Schrödinger equation with zero mass, for which solutions which at leading order are wave packets are shown to exist.

1 Introduction and main results

The problem of the existence of finite-dimensional tori for infinite-dimensional systems, such as nonlinear PDE equations, has been extensively studied in the literature. Up to very recent times, the only available results were confined to the case of one space dimension ($D = 1$). In this context the first results were obtained by Wayne, Kuksin, and Pöschel [24, 20, 21, 22], for the nonlinear Schrödinger equation (NLS) and the nonlinear wave equation (NLW) with Dirichlet boundary conditions, by using KAM techniques. Later on, Craig and Wayne proved similar results, for both Dirichlet and periodic boundary conditions [11], with a rather different method based on the Lyapunov-Schmidt decomposition. The case of periodic boundary condition within the framework of KAM theory was then obtained by Chierchia and You [10]. The case of completely resonant systems, i.e. systems where all eigenvalues of the linear operator are commensurate with each other, was discussed by several authors, and theorems on the existence of periodic solutions for a large measure set of frequencies were obtained by Bourgain [8] for the NLW with periodic boundary conditions, by Gentile, Mastropietro and Procesi [16], and Berti and Bolle [3] for the NLW with Dirichlet boundary conditions, and by Gentile and Procesi [17] for the NLS with Dirichlet boundary conditions. The existence of quasi-periodic solutions for the completely resonant NLW with periodic boundary conditions has been proved by Procesi [23] for a zero-measure set of two-dimensional rotation vectors, by Baldi and Berti [2] for a large measure set of two-dimensional rotation vectors, and by Yuan [25] for a large measure set of – at least three-dimensional – rotation vectors.

Extending the results to higher space dimensions ($D > 1$) introduces a lot of difficulties, mainly due to the high degeneracy of the eigenvalues of the linear operator. The first achievements in this direction were due to Bourgain, and concerned the existence of periodic solutions for NLW [6] and of periodic solutions (also quasi-periodic in $D = 2$) for the NLS [7]. The case of quasi-periodic solutions in arbitrary dimension was solved by Bourgain [9] for the NLS and the NLW. Bourgain's method is based on a Nash-Moser algorithm, which does not imply the linear stability.

A proof of existence and stability of quasi-periodic solutions in high dimension was given by Geng and You using KAM theory. Their result holds for a class of PDE's, which includes the nonlinear beam

equation (NLB) [13] and the NLS with a smoothing nonlinearity [14], with periodic boundary conditions and with nonlinearities which do not depend on the space variable. Both conditions are required in order to ensure a symmetry for the Hamiltonian which simplifies the problem in a remarkable way. Their approach does not extend to the NLS with local nonlinearities – mainly because it requires a "second Melnikov condition" at each iterative KAM step, and such a condition does not appear to be satisfied by the local NLS.

Successively, Eliasson and Kuksin [12], by using KAM techniques, proved the existence and stability of quasi-periodic solutions for the NLS with local nonlinearities. In their paper the main point is indeed to prove that one may impose a second Melnikov condition at each iterative KAM step. However, given a PDE equation, in general (see for instance the case of the NLW in $D > 1$), it can be too hard to impose a second Melnikov condition – even on the unperturbed eigenvalues. Very recently, Yuan [26] proposed a KAM-like approach which does not require the second Melnikov condition, and hence allows to extend the proof of existence to other kinds of equations, including the NLW: with respect to Eliasson and Kuksin's approach the linear stability of the solutions does not follow from the construction.

In both Eliasson and Kuksin's and Yuan's papers Sobolev norms are used to control the regularity of the solutions in the space variables, so that only finite smoothness is found even if the nonlinearity is assumed to be analytic. This is a drawback which does not arise in Bourgain's approach [9], where an exponential decay of the Fourier coefficients is obtained.

Again very recently, Berti and Bolle [5] proved the existence of periodic solutions for PDE systems with eigenvalues of the linear part satisfying rather general separation properties – weaker than those considered in this paper. They use a Nash-Moser algorithm suited for finitely differentiable nonlinearities, already employed in the one-dimensional case [4], and they find solutions belonging to suitable Sobolev classes. By construction, their method looks for a Sobolev regularity, and hence it produces only a finite smoothness even when applied to systems with analytic nonlinearities and with stronger separation properties, as in the cases discussed in this paper. It is very likely that, if we considered analytic nonlinearities and the same weaker separation properties as in [5], we would obtain solutions with only a finite smoothness.

In this paper we revisit the case of periodic solutions with a different method, based on renormalisation group ideas and originally introduced in [15]. We consider analytic nonlinearities, and formulate a general theorem on the existence of periodic solutions in Gevrey class, which emphasises the main assumptions that we need in the proof. From a technical point of view, besides the more abstract formulation – and hence the wider range of application –, the present paper represents an improvement of the renormalisation group method of [18], and allows to considerably simplify the technical aspects of the proof.

For the NLS, with respect to [18] and [14], here we remove the condition for the nonlinearity to be smoothed by a convolution function, so recovering the case of local nonlinearities, as in [7]. Moreover, we obtain results for other equations, including the NLW and the NLB. Finally – and this represents the main novelty of this paper – we discuss cases in which the bifurcation equation is infinite-dimensional, such as the zero-mass NLS and NLB, where the other methods have not been applied so far. In the resonant case the linearised equation has an infinite-dimensional space of periodic solutions with the same period, so that in principle we have at our disposal infinitely many linear solutions with the same period which can be extended to solutions of the nonlinear equation. Indeed we find a denumerable infinity of solutions with the same minimal period even in the presence of the nonlinearity. More precisely, we prove the existence of periodic solutions which at leading order involve an arbitrary finite number of harmonics, and which therefore can be described as distorted wave packets. Solutions of this kind are very natural in the case of completely resonant PDE, where all harmonics are commensurate in the absence of the nonlinearity. An essential ingredient for the existence of such solutions is the particular form of the bifurcation equation: the proof strongly relies on the fact that the leading order of the nonlinearity is cubic and gauge-invariant. Moreover, in order to prove the non-degeneracy of the solutions of the bifurcation equation we need some condition on the higher orders of the nonlinearity. A sufficient condition is that the nonlinearity does not

depend explicitly on the space variables.

The problem of existence of periodic and quasi-periodic solutions in completely resonant systems in higher dimension was already considered by Bourgain in [7], where he constructed quasi-periodic solutions with two frequencies, in $D = 2$, for the NLS with periodic boundary conditions. In the case of Dirichlet boundary conditions, proving the non-degeneracy of the solutions becomes rather involved. We use a combinatorial lemma, proved in [18], and some results in algebraic number theory. With respect to the nonlocal NLS considered in [18], the proof we give here is much simpler, however it has the drawback that a stronger assumption on the nonlinearity is required.

In the remaining part of this section, we give a rigorous description of the PDE systems we shall consider, and a formal statement of the results that we shall prove in the paper. Throughout the paper we shall call a function $F(x,t)$, with $x = (x_1, \dots, x_D) \in \mathbb{R}^D$ and $t \in \mathbb{R}$, even [resp. odd] in x – or even [resp. odd] *tout court* – if it even [resp. odd] in each of its arguments x_i.

Let $\mathbb{S}$ be the D dimensional square $[0,\pi]^D$, and let $\partial\mathbb{S}$ be its boundary. We consider for instance the following class of equations

$$\begin{cases} (\mathrm{i}\partial_t + P(-\Delta) + \mu)\, v = f(x, v, \bar{v}), & (x,t) \in \mathbb{S} \times \mathbb{R}, \\ v(x,t) = 0 & (x,t) \in \partial\mathbb{S} \times \mathbb{R}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, $P(x)$ is a strictly increasing convex C^∞ function with $P(0) = 0$, μ is a real parameter which – we can assume – belongs to some finite interval $(0, \mu_0)$, with $\mu_0 > 0$, and $x \to f(x, v(x,t), \bar{v}(x,t))$ is an analytic function which is super-linear in $v, \bar{v}$ and odd (in x) for odd $v(x,t)$:

$$f(x, v, \bar{v}) = \sum_{r,s \in \mathbb{N}: r+s \geq N+1} a_{r,s}(x)\, v^r \bar{v}^s, \qquad N \geq 1, \tag{1.2}$$

with $a_{r,s}(x)$ even for odd $r+s$ and odd otherwise. We shall look for odd 2π-periodic solutions with periodic boundary conditions in $[-\pi,\pi]^D$.

We require for f in (1.2) to be of the form

$$f(x, v, \bar{v}) = \frac{\partial}{\partial \bar{v}} H(x, v, \bar{v}) + g(x, \bar{v}), \qquad \overline{H(x, v, \bar{v})} = H(x, v, \bar{v}). \tag{1.3}$$

We also consider the class of equations

$$\begin{cases} \left(\partial_{tt} + (P(-\Delta) + \mu)^2\right) v = f(x, v), & (x,t) \in \mathbb{S} \times \mathbb{R}, \\ v(x,t) = 0 & (x,t) \in \partial\mathbb{S} \times \mathbb{R}, \end{cases} \tag{1.4}$$

and finally the wave equation

$$\begin{cases} (\partial_{tt} - \Delta + \mu)\, v = f(x, v), & (x,t) \in \mathbb{S} \times \mathbb{R}, \\ v(x,t) = 0, & \forall (x,t) \in \partial\mathbb{S} \times \mathbb{R}, \end{cases} \tag{1.5}$$

where $f(x,v)$ is of the form (1.2) with s identically zero and $a_r(x) := a_{r,0}(x)$ real (by parity $a_r(x)$ is even for odd r and odd for even r).

We shall consider also (1.1), (1.4) and (1.5) with periodic boundary conditions: in that case, we shall drop the condition for f to be odd.

For all these classes of equations we prove the existence of small periodic solutions with frequency ω close to the linear frequency $\omega_0 = P(D) + \mu$ for (1.1) and (1.4) and $\omega_0 = \sqrt{P(D) + \mu}$ for (1.5), with ω in an appropriate Cantor set of positive measure. We introduce a smallness parameter by rescaling

$$v(x,t) = \varepsilon^{1/N} u(x, \omega t), \qquad \varepsilon > 0, \tag{1.6}$$

with $\omega = P(D) + \mu - \varepsilon$ for (1.1) and (1.4) and $\omega^2 = P(D) + \mu - \varepsilon$ for (1.5).

We shall formulate our results in a more abstract context, by considering the following classes of equations with Dirichlet boundary conditions:

$$\text{(I)} \qquad \begin{cases} \mathbb{D}(\varepsilon)\, u = \varepsilon f(x, u, \bar{u}, \varepsilon^{1/N}), & (x,t) \in \mathbb{S} \times \mathbb{T}, \\ u(x,t) = 0, & (x,t) \in \partial\mathbb{S} \times \mathbb{T}, \end{cases} \tag{1.7a}$$

$$\text{(II)} \qquad \begin{cases} \mathbb{D}(\varepsilon)\, u = \varepsilon f(x, u, \varepsilon^{1/N}), & (x,t) \in \mathbb{S} \times \mathbb{T}, \\ u(x,t) = 0, & (x,t) \in \partial\mathbb{S} \times \mathbb{T}, \end{cases} \tag{1.7b}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{D}(\varepsilon)$ is a linear (possibly integro-)differential wave-like operator with constant coefficients depending on a (fixed once and for all) real parameter ω_0 and on the parameter ε.

We can treat the case of periodic boundary conditions in the same way:

$$\text{(I)} \qquad \mathbb{D}(\varepsilon)\, u = \varepsilon f(x, u, \bar{u}, \varepsilon^{1/N}), \qquad (x,t) \in \mathbb{T}^D \times \mathbb{T}, \tag{1.8a}$$

$$\text{(II)} \qquad \mathbb{D}(\varepsilon)\, u = \varepsilon f(x, u, \varepsilon^{1/N}), \qquad (x,t) \in \mathbb{T}^D \times \mathbb{T}, \tag{1.8b}$$

with the same meaning of the symbols as in (1.7).

In Case (I) we assume that $f(x, u, \bar{u}, \varepsilon^{1/N})$ is a rescaling of a function $f(x, u, \bar{u})$ defined as in (1.2) and satisfying (1.3). In Case (II) we suppose $\mathbb{D}(\varepsilon)$ real and f real for real u, so that it is natural to look for real solutions $u = \bar{u}$.

For $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$ set $\boldsymbol{\nu} = (\nu_0, m)$, with $\nu_0 \in \mathbb{Z}$ and $m = (\nu_1, \ldots, \nu_D) \in \mathbb{Z}^D$ and $|\boldsymbol{\nu}| = |\nu_0| + |m| = |\nu_0| + |\nu_1| + \ldots + |\nu_D|$. For $\mathbf{x} = (t, x) = (t, x_1, \ldots, x_D) \in \mathbb{R}^{D+1}$ set $\boldsymbol{\nu} \cdot \mathbf{x} = \nu_0 t + m \cdot x = \nu_0 t + \nu_1 x_1 + \ldots + \nu_D x_D$. Set also $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}^{D+1}_* = \mathbb{Z}^{D+1} \setminus \{\mathbf{0}\}$. Finally denote by $\delta(i,j)$ the Kronecker delta, i.e. $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ otherwise. Given a finite set $\mathfrak{A}$ we denote by $|\mathfrak{A}|$ the cardinality of the set. Throughout the paper, for $z \in \mathbb{C}$ we denote by $\overline{z}$ the complex conjugate of z.

Since all the results of the paper are local (that is, they concern small amplitude solutions), we shall always assume that the hypotheses below are satisfied for all ε sufficiently small.

Hypothesis 1. ***(Conditions on the linear part).***

1. *$\mathbb{D}(\varepsilon)$ is diagonal in the Fourier basis $\{\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}}\}_{\boldsymbol{\nu} \in \mathbb{Z}^{D+1}}$ with real eigenvalues $\delta_{\boldsymbol{\nu}}(\varepsilon)$ which are C^∞ in both $\boldsymbol{\nu}$ and ε.*
2. *For all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}_*$ one has either $\delta_{\boldsymbol{\nu}}(0) = 0$ or $|\delta_{\boldsymbol{\nu}}(0)| \geq \gamma_0 |\boldsymbol{\nu}|^{-\tau_0}$, for suitable constants $\gamma_0, \tau_0 > 0$.*
3. *For all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}_*$ one has $|\partial_\varepsilon \delta_{\boldsymbol{\nu}}(\varepsilon)| < c_2 |\boldsymbol{\nu}|^{c_0}$ and, if $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < 1/2$, one has $|\partial_\varepsilon \delta_{\boldsymbol{\nu}}(\varepsilon)| > c_1 |\boldsymbol{\nu}|^{c_0}$ as well, for suitable ε-independent constants $c_0, c_1, c_2 > 0$.*
4. *For all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}_*$ such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < 1/2$ one has $|\partial_\varepsilon \partial_{\boldsymbol{\nu}} \delta_{\boldsymbol{\nu}}(\varepsilon)| \leq c_3 |\boldsymbol{\nu}|^{c_0 - 1}$, for a suitable ε-independent constant $c_3 > 0$.*
5. *In case (I) we require that if for some ε and for some $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \mathbb{Z}^{D+1}$ one has $|\delta_{\boldsymbol{\nu}_1}(\varepsilon)|, |\delta_{\boldsymbol{\nu}_2}(\varepsilon)| < 1/2$ then $|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2| \leq |\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2|$.*

We now pass to the equation for the Fourier coefficients. We write

$$u(x,t) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^{D+1}} u_{\boldsymbol{\nu}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}}, \tag{1.9}$$

and introduce the coefficients $u^{\pm}_{\boldsymbol{\nu}}$ by setting $u^+_{\boldsymbol{\nu}} := u_{\boldsymbol{\nu}}$ and $u^-_{\boldsymbol{\nu}} := \overline{u_{\boldsymbol{\nu}}}$. Analogously we define

$$f_{\boldsymbol{\nu}}(\{u\}, \eta) := [f(x, u, \bar{u}, \eta)]_{\boldsymbol{\nu}} = \sum_{r,s \in \mathbb{N} : r+s = N+1} [a_{r,s}(x) u^r \bar{u}^s]_{\boldsymbol{\nu}} + \sum_{r,s \in \mathbb{N} : r+s > N+1} \eta^{r+s-N-1} [a_{r,s}(x) u^r \bar{u}^s]_{\boldsymbol{\nu}}$$

where $\{u\} = \{u_{\boldsymbol{\nu}}^{\sigma}\}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}^{\sigma=\pm}$, $[\cdot]_{\boldsymbol{\nu}}$ denotes the Fourier coefficient with label $\boldsymbol{\nu}$, and we set $f_{\boldsymbol{\nu}}^{+} := f_{\boldsymbol{\nu}}$ and $f_{\boldsymbol{\nu}}^{-} := \overline{f_{\boldsymbol{\nu}}}$. Naturally $f_{\boldsymbol{\nu}}$ depends also on the Fourier coefficients of the functions $a_{r,s}(x)$, which we denote by $a_{r,s,m}$, with $m \in \mathbb{Z}^D$; we set $a_{r,s,m}^{+} := a_{r,s,m}$ and $a_{r,s,m}^{-} := \overline{a_{r,s,m}}$.

Then in Fourier space the equations (1.7) and (1.8) give

$$\delta_{\boldsymbol{\nu}}(\varepsilon)\, u_{\boldsymbol{\nu}}^{\sigma} = \varepsilon f_{\boldsymbol{\nu}}^{\sigma}(\{u\}, \varepsilon^{1/N}), \qquad \boldsymbol{\nu} \in \mathbb{Z}^{D+1}, \qquad \sigma = \pm, \tag{1.10}$$

and in the case of Dirichlet boundary conditions we shall require $u_{\boldsymbol{\nu}} = -u_{S_i(\boldsymbol{\nu})}$ for all $i = 1, \ldots, D$, where $S_i(\boldsymbol{\nu})$ is the linear operator that changes the sign of the i-th component of $\boldsymbol{\nu}$.

Remark 1. *The reality condition on H in (1.3) spells*

$$(s+1)\, a_{s+1,r-1,m}^{-} = r\, a_{r,s,-m}^{+}. \tag{1.11}$$

Moreover, by the analyticity assumption on the nonlinearity, one has $|a_{r,s,m}| \le A_1^{r+s} \mathrm{e}^{-A_2|m|}$ for suitable positive constants A_1 and A_2 independent of r and s.

Remark 2. *We have doubled our equations by considering separately the equations for $u_{\boldsymbol{\nu}}^{+}$ and $u_{\boldsymbol{\nu}}^{-}$ – which clearly must satisfy a compatibility condition. In Case (II) one can work only on $u_{\boldsymbol{\nu}}^{+}$, since $u_{\boldsymbol{\nu}}^{-} = u_{-\boldsymbol{\nu}}^{+}$. In other examples it may be possible to reduce to solutions with $u_{\boldsymbol{\nu}}$ real for all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$, but we found more convenient to introduce the doubled equations in order to deal with the general case.*

Following the standard Lyapunov-Schmidt decomposition scheme we split $\mathbb{Z}^{D+1}$ into two subsets called $\mathfrak{P}$ and $\mathfrak{Q}$ and treat the equations separately. By definition we call $\mathfrak{Q}$ the set of those $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$ such that $\delta_{\boldsymbol{\nu}}(0) = 0$; then we define $\mathfrak{P} = \mathbb{Z}^{D+1} \setminus \mathfrak{Q}$. The equations (1.10) restricted to the $\mathfrak{P}$ and $\mathfrak{Q}$ subset are called respectively the P and Q equations.

Hypothesis 2. ***(Conditions on the Q equation).***

1. *For all $\boldsymbol{\nu} \in \mathfrak{Q}$ one has $\lambda_{\boldsymbol{\nu}}(\varepsilon) := \varepsilon^{-1}\delta_{\boldsymbol{\nu}}(\varepsilon) \ge c > 0$, where c is ε-independent.*

2. *The Q equation at $\varepsilon = 0$,*

$$\lambda_{\boldsymbol{\nu}}(0)\, u_{\boldsymbol{\nu}}^{\sigma} = f_{\boldsymbol{\nu}}^{\sigma}(\{u\}, 0), \qquad \boldsymbol{\nu} \in \mathfrak{Q}, \qquad \sigma = \pm 1,$$

has a non-trivial non-degenerate solution

$$q^{(0)}(x,t) = \sum_{\boldsymbol{\nu}\in\mathfrak{Q}} u_{\boldsymbol{\nu}}^{(0)} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}},$$

where non-degenerate means that the matrix

$$J_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = \lambda_{\boldsymbol{\nu}}(0)\, \delta(\boldsymbol{\nu}, \boldsymbol{\nu}')\, \delta(\sigma, \sigma') - \frac{\partial f_{\boldsymbol{\nu}}^{\sigma}}{\partial u_{\boldsymbol{\nu}'}^{\sigma'}}(\{q^{(0)}\}, 0)$$

is invertible. Moreover one has $|u_{\boldsymbol{\nu}}^{(0)}| \le \Lambda_0 \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}|}$ and $\left|(J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\right| \le \Lambda_0 \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}-\boldsymbol{\nu}'|}$, for suitable constants Λ_0 and λ_0.

Remark 3. *The solution of the* bifurcation equation, *i.e. of the Q equation at $\varepsilon = 0$, could be assumed to be only Gevrey-smooth. Note also that, even when $\mathfrak{Q}$ is infinite-dimensional, the number of non-zero Fourier components of $q^{(0)}(x,t)$ can be finite.*

Definition 1. ***(The sets $\mathfrak{E}_0$, $\mathfrak{D}(\varepsilon)$ and $\mathfrak{D}$).*** *Given $\varepsilon \in \mathfrak{E}_0 := [0, \varepsilon_0]$ we set $\mathfrak{D}(\varepsilon) := \{\boldsymbol{\nu} \in \mathfrak{P} : |\delta_{\boldsymbol{\nu}}(\varepsilon)| < 1/2\}$ and $\mathfrak{D} = \cup_{\varepsilon\in\mathfrak{E}_0}\mathfrak{D}(\varepsilon)$. Finally we call $\mathfrak{R}$ the subset $\mathfrak{P} \setminus \mathfrak{D}$.*

Remark 4. *Note that* $\boldsymbol{\nu} \in \mathfrak{R}$ *means that* $|\delta_{\boldsymbol{\nu}}(\varepsilon)| \geq 1/2$ *for all* $\varepsilon \in \mathfrak{E}_0$.

The following definitions appear (in a slightly different form) in the papers by Bourgain. The notations which we use are those proposed by Berti and Bolle in [5].

Definition 2. ***(The equivalence relation*** $\sim$***).*** *We say that two vectors* $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{D}(\varepsilon)$ *are equivalent, and we write* $\boldsymbol{\nu} \sim \boldsymbol{\nu}'$, *if for* β *small enough the following happens: one has* $|\delta_{\boldsymbol{\nu}}(\varepsilon)|, |\delta_{\boldsymbol{\nu}'}(\varepsilon)| < 1/2$ *and there exists a sequence* $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N\}$ *in* $\mathfrak{D}(\varepsilon)$, *with* $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ *and* $\boldsymbol{\nu}_N = \boldsymbol{\nu}'$, *such that*

$$|\delta_{\boldsymbol{\nu}_k}(\varepsilon)| < \frac{1}{2}, \qquad |\boldsymbol{\nu}_k - \boldsymbol{\nu}_{k+1}| \leq \frac{C_2}{2} \left(|\boldsymbol{\nu}_k| + |\boldsymbol{\nu}_{k+1}|\right)^{\beta}, \qquad k = 1, \ldots, N-1,]$$

where C_2 *is a universal constant. Denote by* $\Delta_j(\varepsilon)$, $j \in \mathbb{N}$, *the equivalence classes with respect to* $\sim$.

Remark 5. *The equivalence relation* $\sim$ *induces a partition of* $\mathfrak{D}(\varepsilon)$ *into disjoint sets* $\{\Delta_j(\varepsilon)\}_{j\in\mathbb{N}}$. *Note also that, if* $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \Delta_j(\varepsilon)$, *then it is not possible that for some* ε' *one has* $\boldsymbol{\nu} \in \Delta_{j_1}(\varepsilon')$ *and* $\boldsymbol{\nu}' \in \Delta_{j_2}(\varepsilon')$ *with* $j_1 \neq j_2$.

Hypothesis 3. ***(Conditions on the set*** $\mathfrak{D}(\varepsilon)$***: separation properties).*** *There exist three* ε*-independent positive constants* α, β, C_1, *with* α *small enough and* $\beta < \alpha$, *such that* $|\Delta_j(\varepsilon)| \leq C_1 p_j^{\alpha}(\varepsilon)$, *where* $p_j(\varepsilon) = \min_{\boldsymbol{\nu}\in\Delta_j(\varepsilon)} |\boldsymbol{\nu}|$, *for all* $j \in \mathbb{N}$.

Remark 6. *Hypothesis 3 implies the following properties:*

$$\begin{aligned} &\mathrm{dist}(\Delta_j(\varepsilon), \Delta_{j'}(\varepsilon)) \geq \frac{C_2}{2} \left(p_j(\varepsilon) + p_{j'}(\varepsilon)\right)^{\beta} \qquad \forall j, j' \in \mathbb{N} \text{ such that } j \neq j', \\ &\mathrm{diam}(\Delta_j(\varepsilon)) \leq C_1 C_2 p_j^{\alpha+\beta}(\varepsilon), \qquad \max_{\boldsymbol{\nu}\in\Delta_j(\varepsilon)} |\boldsymbol{\nu}| \leq 2 p_j(\varepsilon) \qquad \forall j \in \mathbb{N}, \end{aligned}$$

and, furthermore, we can always assume that $2^{c_0-1} C_1 C_2 p_j^{\alpha+\beta} \leq \zeta p_j$, *with* $\zeta c_3 < c_1/4$, *where the constants* c_1 *and* c_3 *are defined in Hypothesis 1.*

Remark 7. *Given* $N > 0$ *and for all* ε *outside a finite set (depending on* N*) the sets* $\Delta_j(\varepsilon) \cap \{\boldsymbol{\nu} : |\boldsymbol{\nu}| \leq N\}$ *are locally constant, namely for all* $\bar{\varepsilon}$ *outside a finite set there exists an interval* $\mathfrak{I}$ *such that* $\bar{\varepsilon} \in \mathfrak{I}$ *with the following property: There exists an* ε*-independent numbering of the sets* $\Delta_j(\varepsilon)$ *contained in* $\{\boldsymbol{\nu} : |\boldsymbol{\nu}| \leq N\}$ *so that* $\Delta_j(\varepsilon) = \Delta_j(\bar{\varepsilon})$ *for all* $\varepsilon \in \mathfrak{I}$.

We can now state our main result.

Theorem 1. *Consider an equation in the class described by (1.7) and (1.8), such that the Hypotheses 1, 2 and 3 hold. There exist a positive constant* ε_0 *and a Cantor set* $\mathfrak{E} \subset [0, \varepsilon_0]$, *such that for all* $\varepsilon \in \mathfrak{E}$ *the equation admits a solution* $u(x,t)$, *which is* 2π*-periodic in time and Gevrey-smooth both in time and in space, and such that*

$$\left| u(x,t) - q^{(0)}(x,t) \right| \leq C \varepsilon^{1/N},$$

uniformly in (x,t). *The set* $\mathfrak{E}$ *has positive Lebesgue measure and*

$$\lim_{\varepsilon \to 0^+} \frac{\mathrm{meas}(\mathfrak{E} \cap [0, \varepsilon])}{\varepsilon} = 1, \tag{1.12}$$

where meas *denotes the Lebesgue measure.*

2 Applications

2.1 Non-resonant equations

Let us prove that the equations (1.1), (1.4), and (1.5) – in particular the NLS, the NLB and the NLW – comply with all the Hypotheses and therefore admit a periodic solution by Theorem 1.

2.1.1 The NLS equation

Theorem 2. *Consider the nonlinear Schrödinger equation in dimension D*

$$\mathrm{i}\partial_t v - \Delta v + \mu v = f(x, v, \bar{v}),$$

with Dirichlet boundary conditions on the square $[0,\pi]^D$, where $\mu \in (0, \mu_0) \subset \mathbb{R}$ and f is given according to (1.2) and (1.3), with $N = 2$, $a_{2,1} = 1$ and $a_{r,s} = 0$ for r, s such that $r + s = 3$ and $(r, s) \neq (2, 1)$, that is $f(x, v, \bar{v}) = |v|^2 v + O(|v|^4)$. There exist a full measure set $\mathfrak{M} \subset (0, \mu_0)$ and a positive constant ε_0 such that the following holds. For all $\mu \in \mathfrak{M}$ there exists a Cantor set $\mathfrak{E}(\mu) \subset [0, \varepsilon_0]$, such that for all $\varepsilon \in \mathfrak{E}(\mu)$ the equation admits a solution $v(x, t)$, which is $2\pi/\omega$-periodic in time and Gevrey-smooth both in time and in space, and such that

$$\left| v(x,t) - \sqrt{\varepsilon} q_0 \mathrm{e}^{\mathrm{i}\omega t} \sin x_1 \ldots \sin x_D \right| \leq C\varepsilon, \qquad \omega = D + \mu - \varepsilon, \qquad |q_0| = \left(\frac{4}{3}\right)^{D/2},$$

uniformly in (x, t). The set $\mathfrak{E} = \mathfrak{E}(\mu)$ has positive Lebesgue measure and satisfies (1.12).

With the notations of Section 1 one has $\delta_{\boldsymbol{\nu}}(\varepsilon) = -\omega n + |m|^2 + \mu$, with $\omega = \omega_0 - \varepsilon$ and $\omega_0 = D + \mu$. Then it is easy to check that all items of Hypothesis 1 are satisfied provided μ is chosen in such a way that $| -\omega_0 n + |m|^2| \geq \gamma_0 |n|^{-\tau_0}$. This is possible for μ in a full measure set; cf. equation (2.1) in [18]. Then Hypothesis 1 holds with $c_0 = c_2 = c_3 = 1$ and $c_1 = 1/\sqrt{1 + 4\omega_0}$.

The subset $\mathfrak{Q}$ is defined as $\mathfrak{Q} := \{(n, m) \in \mathbb{Z}^{1+D} : n = 1,\ |m_i| = 1\ \forall i = 1, \ldots D\}$, and one can assume take q_0 to be real, so that, by the Dirichlet boundary conditions, $\mathfrak{Q}$ is in fact one-dimensional, and $u_{n,m} = \pm q_0$ for all $(n, m) \in \mathfrak{Q}$. The leading order of the Q equation is explicitly studied in [18], where it is proved that Hypothesis 2 is satisfied.

Finally, Hypothesis 3 has been proven by Bourgain [7] (see also Appendix A6 in [18]).

Of course, Theorem 2 refers to solutions with $m = (1, 1, \ldots, 1)$, but it easily extends to solutions which continue other harmonics of the linear equation; see comments in [18].

Also, the condition on the nonlinearity can be weakened. In general N can be any integer $N > 1$, and no other conditions must be assumed on the functions $a_{r,s}(x)$ beyond those mentioned after (1.2). In that case (for simplicity we consider the same solution of the linear equation as in Theorem 2), the leading order of the Q equation becomes $q_0 = \mathrm{sign}(\varepsilon) A_0 q_0^N$ (again by taking for simplicity's sake q_0 to be real), where A_0 is a constant depending on the nonlinearity. If A_0 is non-zero, this surely has a non-trivial non-degenerate solution q_0 either for positive or negative values of ε. In general the non-degeneracy condition in item 2 of Hypothesis 2 has to be verified case by case by computing A_0.

2.1.2 The NLW equation

Theorem 3. *Consider the nonlinear wave equation in dimension D*

$$\partial_{tt} v - \Delta v + \mu v = f(x, v),$$

with Dirichlet boundary conditions on the square $[0,\pi]^D$, where $\mu \in (0, \mu_0) \subset \mathbb{R}$ and f is given according to (1.2), with $s = 0$, $N = 2$, $a_{3,0} = 1$, that is $f(x, v) = v^3 + O(v^4)$. There exist a full measure set $\mathfrak{M} \subset (0, \mu_0)$ and a positive constant ε_0 such that the following holds. For all $\mu \in \mathfrak{M}$ there exists a Cantor set $\mathfrak{E}(\mu) \subset [0, \varepsilon_0]$, such that for all $\varepsilon \in \mathfrak{E}(\mu)$ the equation admits a solution $v(x, t)$, which is $2\pi/\omega$-periodic in time and Gevrey-smooth both in time and in space, and such that

$$\left| v(x,t) - q_0 \sqrt{\varepsilon} \cos \omega t \sin x_1 \ldots \sin x_D \right| \leq C\varepsilon, \qquad \omega = \sqrt{D + \mu - \varepsilon}, \qquad q_0 = \left(\frac{4}{3}\right)^{(D+1)/2},$$

uniformly in (x, t). The set $\mathfrak{E} = \mathfrak{E}(\mu)$ has positive Lebesgue measure and satisfies (1.12).

In that case one has $\delta_{\boldsymbol{\nu}}(\varepsilon) = -\omega^2 n^2 + |m|^2 + \mu$, with $\omega^2 = \omega_0^2 - \varepsilon$ and $\omega_0^2 = D^2 + \mu$. Once more, it is easy to check that Hypothesis 1 is satisfied provided μ is chosen in a full measure set, with $c_0 = c_2 = c_3 = 1$ and $c_1 = 1/(1 + 4\omega_0^2)$.

The subset $\mathfrak{Q}$ is given by $\mathfrak{Q} := \{(n, m) \in \mathbb{Z}^{1+D} : n = \pm 1,\ |m_i| = 1\ \forall i = 1, \dots D\}$, and, if one chooses to look for solutions that are even in time, then $\mathfrak{Q}$ is one-dimensional. The Q equation at $\varepsilon = 0$ can be discussed as in the case of the nonlinear Schrödinger equation. For instance for f as in the statement of Theorem 3 the non-degeneracy in item 2 of Hypothesis 2 can be explicitly verified. Again, the analysis easily extends to more general situations, under the assumption that the Q equation at $\varepsilon = 0$ admits a non-degenerate solution. For a fixed nonlinearity, this can be easily checked with a simple computation.

Hypothesis 3 has been verified by Bourgain [6], under some strong conditions on ω. Recently the same separation estimates have been proved by Berti and Bolle [5], by only requiring that ω^2 be Diophantine.

2.1.3 Other equations

Of course, the separation properties for the NLS equation imply similar separation also for the nonlinear beam (NLB) equation

$$\partial_{tt} v + (\Delta + \mu)^2\ v = f(x, v),$$

and in that case we can also consider nonlinearities with one or two space derivatives.

As in the previous cases one restricts μ to some full measure set, and Hypothesis 1 holds with $c_0 = c_3 = 2$, $c_2 = 1$ and $c_1 = 1/\sqrt{1 + 2\omega_0}$. This implies that the subset $\mathfrak{Q}$ is one-dimensional, provided we look for real solutions which are even in time.

The same kind of arguments holds for all equations of the form (1.1) and (1.4). The separation of the points $(m, |m|^2)$ in $\mathbb{Z}^{D+1}$ implies, by convexity, also the separation of $(m, P(|m|^2))$, with $P(x)$ defined after (1.1).

2.2 Completely resonant equations

Here we describe an application to completely resonant NLS and NLB equations, namely equations (1.1) and (1.4) with $P(x) = x$ and $\mu = 0$, and with Dirichlet boundary conditions (the case of periodic boundary conditions is easier for fully resonant equations). Since the equation is completely resonant we need some assumption on the nonlinearity in order to comply with Hypothesis 2. We set $f(x, v, \bar{v}) = |v|^2 v$ for the NLS and $f(x, v) = v^3$ for the NLB (the NLB falls in case (II) and we look for real solutions), but our proofs extend easily to deal with higher order corrections which are odd and do not depend explicitly on the space variables. In the case of the NLS we say that the leading term of the nonlinearity is cubic and gauge-invariant.[1]

The validity of Hypothesis 1 can be discussed as in the non-resonant equations of Subsections 2.1. The separation properties (Hypothesis 3) do not change in the presence of a mass term, and they have been already discussed in the non-resonant examples of Subsection 2.1. Thus, we only need to prove the non-degeneracy of the solution of the Q equation. Since the nonlinearity does not depend explicitely on x we look for solutions such that $u_{\boldsymbol{\nu}} \in \mathbb{R}$. We follow closely [18], but we set $\omega_0 = 1$. This is done for purely notational reasons, and is due to the fact that a trivial rescaling of time allows us to put $\omega_0 = 1$.

2.2.1 The NLS equation

The subset $\mathfrak{Q}$ is infinite-dimensional, i.e. $\mathfrak{Q} := \{(n, m) \in \mathbb{N} \times \mathbb{Z}^D :\ n = |m|^2\}$. We set $u_{(n,m)} = q_m = a_m + O(\varepsilon^{1/2})$ for $(n, m) \in \mathfrak{Q}$ and restrict our attention to the case $q_m \in \mathbb{R}$. At leading order, the Q

[1] i.e. the equation up to the third order is invariant under the transformation $v \to v^{i\alpha}$ for any $\alpha \in \mathbb{R}$.

equation is (cf. [18])

$$|m|^2 a_m = \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle = 0}} a_{m_1} a_{m_2} a_{m_3}. \tag{2.1}$$

Note that in the case of [18], the left hand side of (2.1) was $|m|^{2+2s} D^{-1} a_m$, with s a free parameter; then (2.1) is recovered by setting $s = 0$ and rescaling by $1/\sqrt{D}$ the coefficients q_m.

By Lemma 17 of [18] – which holds for all values of s –, for each $N_0 \geq 1$ there exist infinitely many finite sets $\mathcal{M}_+ \subset \mathbb{Z}^D_+$ with N_0 elements such that equation (2.1) admits the solution (due to the Dirichlet boundary conditions we describe the solution in $\mathbb{Z}^D_+$)

$$a_m = \begin{cases} 0, & m \in \mathbb{Z}^D_+ \setminus \mathcal{M}_+ \\ \sqrt{\dfrac{1}{2^{D+1} - 3^D}\Big(|m|^2 - c_1 \displaystyle\sum_{m' \in \mathcal{M}_+} |m'|^2\Big)}, & m \in \mathcal{M}_+, \end{cases}$$

with $c_1 = 2^{D+1}/(2^{D+1}(N_0 - 1) + 3^D)$. The set $\mathcal{M}_+$ defines a matrix J on $\mathbb{Z}^D$ such that

$$(JQ)_m = |m|^2 - 2 \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle = 0}} Q_{m_1} a_{m_2} a_{m_3} - 2 \sum_{\substack{m_1 > m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle = 0}} a_{m_1} a_{m_2} Q_{m_3}, \tag{2.2}$$

where $m_1 > m_2$ refers, say, to lexicographic ordering of $\mathbb{Z}^D$; see in particular equations (8.5) and (8.7) of [18].

Moreover we know (Lemma 18 of [18]) that the matrix J is block-diagonal with blocks of size depending only on N_0, D: we denote by $K(N_0, D)$ the bound on such a size. Whatever the block structure, the matrix J has the form $\mathrm{diag}(|m|^2) + 2T$ where all the entries of T are linear combinations of terms $q_{m_i} q_{m_j}$ with integer coefficients. If we multiply J by $z := (2^{D+1} - 3^D)(2^{D+1}(N_0 - 1) + 3^D)$ – which is odd –, we obtain a matrix $J' := \mathrm{diag}(z|m|^2) + 2T'$, where all the entries of T' are integral linear combinations of the square roots of a finite number of integers. Let us call the prime factors of such integers $p_0 = 1, p_1, p_2, \ldots$.

Definition 3. ***(The lattice $\mathbb{Z}^D_1$).*** *Let $\mathbb{Z}^D_1 := (1, 0, \ldots, 0) + 2\mathbb{Z}^D$ be the affine lattice of integer vectors such that the first component is odd and the others even. Let $\mathbb{Z}^D_{1,+}$ be its intersection with $\mathbb{Z}^D_+$. Of course, for all $m \in \mathbb{Z}^D_1$ one has $|m|^2$ odd.*

Since we are working with odd nonlinearities which do not depend explicitly on the space variables we look for solutions such that $u_{n,m} = 0$ if $m \notin \mathbb{Z}^D_1$.

Let $1, p_1, \ldots, p_k$ be prime numbers (as above), and let $a_1, \ldots, a_K$ be the set of all products of square roots of different numbers p_i, i.e. $a_1 = 1$, $a_2 = \sqrt{p_1}$, $a_3 = \sqrt{p_1 p_2}$, etc. It is clear that the set of integral linear combinations of a_i is a ring (of algebraic integers). We denote it by $\mathfrak{a}$. The following Lemma is a simple consequence of Galois theory [1]. For completeness, the proof is given in Appendix A.

Lemma 1. *The numbers a_i are linearly independent over the rationals.*

Immediately we have the following corollary (I denotes the identity).

Corollary 1. *In $\mathfrak{a}$ consider $2\mathfrak{a}$, i.e. the set of linear combinations with even coefficients.*

- *$2\mathfrak{a}$ is a proper ideal, and the quotient ring $\mathfrak{a}/2\mathfrak{a}$ is thus a non-zero ring.*
- *if a matrix M with entries in $\mathfrak{a}$ is such that $M - I$ has all entries in $2\mathfrak{a}$, then M is invertible.*

The point of Corollary 1 is that the determinant of $M = I + 2M'$, with the entries of M' in $\mathfrak{a}$, is $1 + 2\alpha$, with $\alpha \in \mathfrak{a}$. Hence, by Lemma 1, $2\alpha \neq \pm 1$.

Lemma 2. *For all N_0 and for all $\mathcal{M}_+ \subset \mathbb{Z}^D_{1,+}$ the matrix J defined by $\mathcal{M}_+$ is invertible. Its inverse is a block matrix with blocks of dimension depending only on N_0, D so that for some appropriate C one has $(J^{-1})_{m,m'} \leq C$ if $|m - m'| \leq K(N_0, D)$, while $(J^{-1})_{m,m'} = 0$ otherwise.*

Proof. We use Corollary 1, the fact that the matrix J' has entries in $\mathfrak{a}$ and the fact that $z|m|^2$ is odd for all $m \in \mathbb{Z}^D_{1,+}$. ∎

Now, we can state our result on the completely resonant NLS.

Theorem 4. *Consider the nonlinear Schrödinger equation in dimension D*

$$\mathrm{i}\partial_t v - \Delta v = f(v, \bar{v}),$$

with Dirichlet boundary conditions on the square $[0, \pi]^D$, where f is given according to (1.2) and (1.3), with $N = 2$, $a_{2,1} = 1$, $a_{r,s} = 0$ for r, s such that $r + s = 3$ and $(r, s) \neq (2, 1)$, and $a_{r,s}(x)$ independent of x for $r + s > 3$ (so that in particular $a_{r,s} = 0$ for even $r + s$). For any $N_0 \geq 1$ there exist sets $\mathcal{M}_+$ of N_0 vectors in $\mathbb{Z}^D_+$ and real amplitudes $\{a_m\}_{m\in\mathcal{M}_+}$ such that the following holds. There exist a positive constant ε_0 and a Cantor set $\mathfrak{E} \subset [0, \varepsilon_0]$, such that for all $\varepsilon \in \mathfrak{E}$ the equation admits a solution $v(x, t)$, which is $2\pi/\omega$-periodic in time and Gevrey-smooth both in time and in space, and such that, setting

$$q_0(x, t) = (2\mathrm{i})^D \sum_{m\in\mathcal{M}_+} a_m \mathrm{e}^{\mathrm{i}|m|^2 t} \sin m_1 x_1 \dots \sin m_D x_D, \qquad \omega = 1 - \varepsilon, \tag{2.3}$$

one has

$$\left|v(x, t) - \sqrt{\varepsilon} q_0(x, \omega t)\right| \leq C\varepsilon,$$

uniformly in (x, t). The set $\mathfrak{E}$ has positive Lebesgue measure and satisfies (1.12).

2.2.2 The beam equation

We set $\omega^2 = \omega_0^2 - \varepsilon = 1 - \varepsilon$ (recall that we are assuming $\omega_0 = 1$ by a suitable time rescaling). The subset $\mathfrak{Q}$ is given by $\mathfrak{Q} := \{(n, m) \in \mathbb{N} \times \mathbb{Z}^D : |n| = |m|^2\}$. We set $u_{n,m} = q^+_m$ for $n = |m|^2$ and $u_{n,m} = q^-_m$ for $n = -|m|^2$. We can require that $q^+_m = q^-_m \equiv q_m$ for all m (we obtain a solution which is even in time). Since we look for real solutions, this implies that $q_m \in \mathbb{R}$ if D is even and $q_m \in \mathrm{i}\mathbb{R}$ if D is odd. Since the nonlinearity does not depend explicitly on x, we can look for solutions $u_{n,m}$ such that $m \in \mathbb{Z}^D_1$ (see Definition 3).

Finally the separation properties of the small divisors do not depend on the presence of the mass term, so that we only need to prove the existence and non-degeneracy of the solutions of the bifurcation equation.

The Q equation at leading order is

$$|m|^4 a_m = (-1)^D \sum_{\substack{m_1+m_2+m_3=m \\ \pm|m_1|^2\pm|m_2|^2\pm|m_3|^2=\pm|m|^2}} a_{m_1} a_{m_2} a_{m_3},$$

where we have set $|q_m| = a_m + O(\varepsilon^{1/2})$.

Lemma 3. *The condition $\pm|m_1|^2 \pm |m_2|^2 \pm |m_3|^2 = \pm|m|^2$, for $m_i, m \in \mathbb{Z}^D_1$, is equivalent to $\langle m_1 + m_3, m_2 + m_3\rangle = 0$.*

Proof. The condition $|m_1|^2+|m_2|^2+|m_3|^2=(m_1+m_2+m_3)^2$ is equivalent to $\langle m_1, m_2+m_3\rangle+\langle m_2, m_3\rangle=0$, which is impossible since the left hand side is an odd integer. The same happens with the condition $|m_1|^2-|m_2|^2-|m_3|^2=(m_1+m_2+m_3)^2$. Thus, we are left with $|m_1|^2+|m_2|^2-|m_3|^2=(m_1+m_2+m_3)^2$, which implies $\langle m_1+m_3, m_2+m_3\rangle=0$. ■

Lemma 3 implies that the bifurcation equation, restricted to $\mathbb{Z}_1^D$, is identical to that of a smoothing NLS with $s=2$; cf. [18]. Indeed by recalling that $q_m=(-1)^D q_{-m}$ one has

$$|m|^4 a_m = \sum_{\substack{m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle=0}} a_{m_1} a_{m_2} a_{m_3}. \tag{2.4}$$

Then we can repeat the arguments of the previous subsection. By Lemma 17 of [18] – which holds for all values of s – for each $N_0 \geq 1$ there exist infinitely many finite sets $\mathcal{M}_+ \subset \mathbb{Z}_{1,+}^D$ with N_0 elements such that the equation (2.4) has the solution

$$a_m = \begin{cases} 0, & m \in \mathbb{Z}_+^D \setminus \mathcal{M}_+ \\ \sqrt{\dfrac{1}{2^{D+1}-3^D}\Big(|m|^4 - c_1 \displaystyle\sum_{m' \in \mathcal{M}_+} |m'|^4\Big)}, & m \in \mathcal{M}_+, \end{cases}$$

with $c_1 = 2^{D+1}/(2^{D+1}(N_0-1)+3^D)$.

The matrix J is defined as in (2.2), only with $|m|^4$ on the diagonal. We know (Lemma 18 of [18] does not depend on the values of s) that the matrix J is block-diagonal with blocks of size bounded by $K(N_0, D)$ (defined as in subsection 2.2.1). Whatever the block structure, the matrix J has the form $\mathrm{diag}(|m|^4)+2T$, where all the entries of T are linear combinations of terms $a_{m_i}a_{m_j}$ with integer coefficients. If we multiply J by $z := (2^{D+1}-3^D)(2^{D+1}(N_0-1)+3^D)$ – which is odd –, we obtain a matrix $J' := \mathrm{diag}(z|m|^4)+2T'$, where all the entries of T' are linear combinations of the square roots of a finite number of integers; finally $z|m|^4$ is clearly odd and we can apply Lemma 1 to obtain the analogous of Lemma 2. Thus, a theorem analogous to Theorem 4 is obtained, with $q_0(x,t)$ in (2.3) replaced with

$$q_0(x,t) = 2^{D+1} \sum_{m \in \mathcal{M}_+} a_m \cos|m|^2 t \, \sin m_1 x_1 \dots \sin m_D x_D, \qquad \omega^2 = 1-\varepsilon.$$

We leave the formulation to the reader.

3 Technical set-up and propositions

3.1 Renormalised P-Q equations

Group the equations (1.10) for $\boldsymbol{\nu} \in \mathfrak{O}$ as a matrix equation. Setting

$$U = \{u_{\boldsymbol{\nu}}^\sigma\}_{\boldsymbol{\nu}\in\mathfrak{O}}^{\sigma=\pm}, \quad V = \{u_{\boldsymbol{\nu}}^\sigma\}_{\boldsymbol{\nu}\in\mathfrak{R}}^{\sigma=\pm}, \quad Q = \{u_{\boldsymbol{\nu}}^\sigma\}_{\boldsymbol{\nu}\in\mathfrak{Q}}^{\sigma=\pm}, \quad F = \{f_{\boldsymbol{\nu}}^\sigma\}_{\boldsymbol{\nu}\in\mathfrak{O}}^{\sigma=\pm}, \quad \mathbb{D}(\varepsilon) = \mathrm{diag}\,\{\delta_{\boldsymbol{\nu}}(\varepsilon)\}_{\boldsymbol{\nu}\in\mathfrak{O}}^{\sigma=\pm}, \tag{3.1}$$

the P equations spell

$$\begin{cases} \mathbb{D}(\varepsilon)\, U = \varepsilon F(U,V,Q,\varepsilon^{1/N}), \\ u_{\boldsymbol{\nu}}^\sigma = \varepsilon \delta_{\boldsymbol{\nu}}^{-1}(\varepsilon)\, f_{\boldsymbol{\nu}}^\sigma(U,V,Q,\varepsilon^{1/N}), \quad \boldsymbol{\nu} \in \mathfrak{R}, \end{cases} \tag{3.2}$$

with a reordering of the arguments of the coefficients $f_{\boldsymbol{\nu}}^\sigma$.

We want to introduce an appropriate "correction" to the left hand side of (3.2). We shall consider self-adjoint matrices $\widehat{M}(\varepsilon) := \{\widehat{M}_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon)\}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{O}}^{\sigma,\sigma'=\pm}$, which for each fixed ε are block-diagonal on the sets

$\Delta_j(\varepsilon)$ (cf. Definition 2), namely $\widehat{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) \neq 0$ can hold only if $\boldsymbol{\nu},\boldsymbol{\nu}' \in \Delta_j(\varepsilon)$ for some j. Moreover we require for $\widehat{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)$ to depend smoothly on ε, at least in a large measure set.

We shall first introduce the self-adjoint matrices $\widehat{M}$ as independent parameters, and eventually we shall manage to fix them as functions of the parameter ε. Note that in order to have $u^+_{\boldsymbol{\nu}} = \overline{u^-_{\boldsymbol{\nu}}}$ we must require that $\widehat{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = \widehat{M}^{-\sigma',-\sigma}_{\boldsymbol{\nu}',\boldsymbol{\nu}}$.

Definition 4. ***(The set $\mathfrak{G}$ and the matrix $\widehat{\chi}_1$).*** *Call $\mathfrak{G} = \{1/4 > \bar{\gamma} > 0 : ||\delta_{\boldsymbol{\nu}}(0)| - \bar{\gamma}| \geq \bar{\gamma}_0/|\boldsymbol{\nu}|^{\bar{\tau}_0}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}_*\}$, for suitable constants $\bar{\gamma}_0, \bar{\tau}_0 > 0$. For $\bar{\gamma} \in \mathfrak{G}$, we introduce the step function $\bar{\chi}_1(x)$ such that $\bar{\chi}_1(x) = 0$ if $|x| \geq \bar{\gamma}$ and $\bar{\chi}_1(x) = 1$ if $|x| < \bar{\gamma}$, and set $\bar{\chi}_0(x) = 1 - \bar{\chi}_1(x)$. We then introduce the (ε-dependent) diagonal matrices $\widehat{\chi}_1 = \mathrm{diag}\{\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{D}}$ and $\widehat{\chi}_0 = \mathrm{diag}\{\bar{\chi}_0(\delta_{\boldsymbol{\nu}}(\varepsilon))\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{D}}$.*

Remark 8. *One has $\mathfrak{G} \neq \emptyset$. Moreover, for any interval $\mathfrak{U} \subset (0, 1/4)$, the relative measure of the set $\mathfrak{U} \cap \mathfrak{G}$ tends to 1 as $\bar{\gamma}_0$ tends to 0, provided $\bar{\tau}_0$ is large enough*

Remark 9. *Note that $\widehat{\chi}_1^2 = \widehat{\chi}_1$ and $\widehat{\chi}_1\widehat{\chi}_0 = 0$, with 0 the null matrix.*

Definition 5. ***(Resonant sets).*** *A set $\mathcal{N} = \{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\} \subset \mathfrak{D}$ is* resonant *if there exists $\varepsilon \in \mathfrak{E}_0$ and $j \in \mathbb{N}$ such that $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m \in \Delta_j(\varepsilon)$. A resonant set $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\}$ with $m = 2$ will be called a* resonant pair. *Given a resonant set $\mathcal{N} = \{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\}$ we call $\mathcal{C}_{\mathcal{N}}$ the set of all $\boldsymbol{\nu} \in \mathfrak{D}$ such that $\mathcal{N} \cup \{\boldsymbol{\nu}\}$ is still a resonant set. Finally set $\overline{\mathcal{C}}_{\mathcal{N}}(\varepsilon) := \{\boldsymbol{\nu}' \in \mathcal{C}_{\mathcal{N}} : |\delta_{\boldsymbol{\nu}'}(\varepsilon)| < \bar{\gamma}\}$.*

Define the *renormalised P equation* as

$$\begin{cases} \left(\mathbb{D}(\varepsilon) + \widehat{M}\right) U = \eta^N F(U, V, Q, \eta) + L\, U, \\ u^\sigma_{\boldsymbol{\nu}} = \eta^N\, \delta^{-1}_{\boldsymbol{\nu}}(\varepsilon)\, f^\sigma_{\boldsymbol{\nu}}(U, V, Q, \eta), \qquad\qquad \boldsymbol{\nu} \in \mathfrak{R}, \end{cases} \tag{3.3}$$

with $\widehat{M} = \widehat{\chi}_1 M \widehat{\chi}_1$, where η is a real parameter, while $M = \{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\}^{\sigma,\sigma'=\pm}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{D}}$ and $L = \{L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\}^{\sigma,\sigma'=\pm}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{D}}$ are self-adjoint matrices of free parameters with the properties:

1. $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = 0$ if $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ is not a resonant pair.
2. $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = M^{-\sigma',-\sigma}_{\boldsymbol{\nu}',\boldsymbol{\nu}}$ and $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = L^{-\sigma',-\sigma}_{\boldsymbol{\nu}',\boldsymbol{\nu}}$.

The *renormalised Q equation* is defined as

$$u^\sigma_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\nu}\in\mathfrak{Q}} \sum_{\sigma'=\pm} (J^{-1})^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} f^{\sigma'}_{\boldsymbol{\nu}'}(U, V, Q, \eta), \qquad \boldsymbol{\nu} \in \mathfrak{Q}. \tag{3.4}$$

The parameter η and the *counterterms* L will have to satisfy eventually the identities (*compatibility equation*)

$$\eta = \varepsilon^{1/N}, \qquad \widehat{M} = L. \tag{3.5}$$

We proceed in the following way: first we solve the renormalised P and Q equations (3.3) and (3.4), then we impose the compatibility equation (3.5).

3.2 Matrix spaces

Here we introduce some notations and properties that we shall need in the following.

Definition 6. ***(The Banach space $\mathcal{B}_\kappa$).*** *We consider the space of infinite-dimensional self-adjoint matrices $\{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\}^{\sigma,\sigma'=\pm}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{O}}$ such that $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}=0$ if $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$ is not resonant. For $\rho,\kappa>0$ we equip such a space with the norm*

$$|M|_\kappa := \sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{O}} \sup_{\sigma,\sigma'=\pm} \left| M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \mathrm{e}^{\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho},$$

so obtaining a Banach space that we call $\mathcal{B}_\kappa$. For L a linear operator on $\mathcal{B}_\kappa$ define the operator norm

$$|L|_{\mathrm{op}} = \sup_{M\in\mathcal{B}_\kappa} \frac{|LM|_\kappa}{|M|_\kappa}.$$

Definition 7. ***(Matrix norms).*** *Let A be a $d\times d$ self-adjoint matrix, and denote with $A(a,b)$ and $\lambda^{(a)}(A)$ its entries and its eigenvalues, respectively. We define the norms*

$$|A|_\infty := \max_{1\le a,b\le d} |A(a,b)|, \qquad \|A\| := \frac{1}{\sqrt{d}}\sqrt{\mathrm{tr}(A^2)}, \qquad \|A\|_2 := \max_{|x|_2\le 1} |Ax|_2\,,$$

where, given a vector $x\in\mathbb{R}^d$, we denote by $|x|_2$ its Euclidean norm.

Lemma 4. *Given $d\times d$ self-adjoint matrix A, the following properties hold.*

1. *The norm $\|A\|$ depends smoothly on the coefficients $A(a,b)$.*
2. *One has $\|A\|/\sqrt{d} \le |A|_\infty \le \sqrt{d}\|A\|$.*
3. *One has $\max_{1\le a\le d} |\lambda^{(a)}(A)|/\sqrt{d} \le \|A\| \le \max_{1\le a\le d} |\lambda^{(a)}(A)|$.*
4. *For invertible A one has $\partial_{A(a,b)} A^{-1}(a',b') = -A^{-1}(a',a)\,A^{-1}(b,b')$ and $\partial_{A(a,b)}\|A\| = A(a,b)/d\|A\|$.*

Here and henceforth we shall write $A=\mathbb{D}(\varepsilon)+\widehat{M}$ in (3.3).

Definition 8. ***(Small divisors).*** *For $\boldsymbol{\nu}\in\mathfrak{O}$ define $A^{\boldsymbol{\nu}}(\varepsilon)$ as the matrix with entries $\bar\chi_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\,A^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}$ such that $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$ and $\sigma_1,\sigma_2=\pm$. If $|\delta_{\boldsymbol{\nu}}(\varepsilon)|<\bar\gamma$, define also $d^{\boldsymbol{\nu}}(\varepsilon):=2|\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)|$ and $p_{\boldsymbol{\nu}}(\varepsilon)=\min\{|\boldsymbol{\nu}'|:\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)\}$. For real positive ξ, define the* small divisor

$$x_{\boldsymbol{\nu}}(\varepsilon) := \frac{1}{p^{\xi}_{\boldsymbol{\nu}}(\varepsilon)} \left\|(A^{\boldsymbol{\nu}}(\varepsilon))^{-1}\right\|^{-1},$$

if A is invertible, and set $x_{\boldsymbol{\nu}}(\varepsilon)=0$ if A is not invertible.

Remark 10. *Note that for $\boldsymbol{\nu}\in\Delta_j(\varepsilon)$ one has $p_{\boldsymbol{\nu}}(\varepsilon)=p_j(\varepsilon)$, $d_{\boldsymbol{\nu}}(\varepsilon)\le 2|\Delta_j(\varepsilon)|$, and $A^{\boldsymbol{\nu}}(\varepsilon)=A^{\boldsymbol{\nu}'}(\varepsilon)$ for all $\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. This shows that $d_{\boldsymbol{\nu}}(\varepsilon)$, $x_{\boldsymbol{\nu}}(\varepsilon)$ and $p_{\boldsymbol{\nu}}(\varepsilon)$ are the same for all $\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. Note also that, if $\boldsymbol{\nu}\in\Delta_j(\varepsilon)$ for some $j\in\mathbb{N}$, then one has $\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)=\{\boldsymbol{\nu}'\in\Delta_j(\varepsilon):|\delta_{\boldsymbol{\nu}'}(\varepsilon)|<\bar\gamma\}$. Hypothesis 3 implies $d_{\boldsymbol{\nu}}(\varepsilon)\le 2C_1p^\alpha_{\boldsymbol{\nu}}(\varepsilon)$.*

Definition 9. ***(The sets $\mathfrak{D}_0$, $\mathfrak{D}_1(\gamma)$, $\mathfrak{D}_2(\gamma)$, and $\mathfrak{D}(\gamma)$).*** *We define $\mathfrak{D}_0=\{(\varepsilon,M):\varepsilon\in\mathfrak{E}_0,\ |M|_\kappa\le C_0\varepsilon_0\}$, for a suitable positive constant C_0, and, for fixed $\tau,\tau_1>0$ and $\gamma<\bar\gamma$, we set $\mathfrak{D}_1(\gamma)=\{(\varepsilon,M)\in\mathfrak{D}_0: x_{\boldsymbol{\nu}}\ge\gamma/p^\tau_{\boldsymbol{\nu}}(\varepsilon)$ for all $j\in\mathbb{N}\}$, $\mathfrak{D}_2(\gamma)=\{(\varepsilon,M)\in\mathfrak{D}_0:||\delta_{\boldsymbol{\nu}}(\varepsilon)|-\bar\gamma|\ge\gamma/|\boldsymbol{\nu}|^{\tau_1}$ for all $\boldsymbol{\nu}\in\mathfrak{O}\}$, and $\mathfrak{D}(\gamma)=\mathfrak{D}_1(\gamma)\cap\mathfrak{D}_2(\gamma)$.*

Definition 10. ***(The sets $\mathcal{I}_{\mathcal{N}}(\gamma)$ and $\overline{\mathcal{I}}_{\mathcal{N}}(\gamma)$).*** *Given a resonant set $\mathcal{N}$ we define $\overline{\mathcal{I}}_{\mathcal{N}}(\gamma):=\{\varepsilon\in\mathfrak{E}_0:\exists\boldsymbol{\nu}\in\mathcal{C}_{\mathcal{N}}$ such that $||\delta_{\boldsymbol{\nu}}(\varepsilon)|-\bar\gamma|<\gamma|\boldsymbol{\nu}|^{-\tau_1}\}$, and set $\mathcal{I}_{\mathcal{N}}(\gamma):=\{(\varepsilon,M)\in\mathfrak{D}_0:\varepsilon\in\overline{\mathcal{I}}_{\mathcal{N}}(\gamma)\}$.*

3.3 Main propositions

We state the propositions which represent our main technical results. Theorem 1 is an immediate consequence of Propositions 1 and 2 below.

Proposition 1. *There exist positive constants $K_0, K_1, \kappa, \rho, \eta_0$ such that the following holds true. For $(\varepsilon, M) \in \mathfrak{D}(\gamma)$, there exists a matrix $L(\eta, \varepsilon, M) \in \mathcal{B}_\kappa$, such that the following holds.*

1. *For each ε the matrix $L(\eta, \varepsilon, M)$ is block-diagonal so as to satisfy $L(\eta, \varepsilon, M) = \widehat{\chi}_1 L(\eta, \varepsilon, M) \widehat{\chi}_1$.*
2. *There exists a unique solution $u^\sigma_{\boldsymbol{\nu}}(\eta, M, \varepsilon)$, with $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$, of equations (3.3) and (3.4), which is analytic in η for $|\eta| \le \eta_0$, and such that for all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$ and $\sigma = \pm$*
$$|u^\sigma_{\boldsymbol{\nu}}(\eta, M, \varepsilon)| \le |\eta| \, K_0 \mathrm{e}^{-\kappa |\boldsymbol{\nu}|^{1/2}}.$$
3. *The matrix elements $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ are analytic in η for $|\eta| \le \eta_0$, and uniformly bounded for $(\varepsilon, M) \in \mathfrak{D}(\gamma)$ as*
$$|L(\eta, \varepsilon, M)|_\kappa \le |\eta|^N K_0.$$
4. *The functions $u^\sigma_{\boldsymbol{\nu}}(\eta, \varepsilon, M)$ can be extended on the set $\mathfrak{D}_0$ to C^1 functions $u^{E\,\sigma}_{\boldsymbol{\nu}}(\eta, \varepsilon, M)$, and the matrix elements $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ can be extended on the set $\mathfrak{D}_0 \setminus \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ to C^1 functions $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$, such that $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M) = L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ and $u^{E\,\sigma}_{\boldsymbol{\nu}}(\eta, \varepsilon, M) = u^\sigma_{\boldsymbol{\nu}}(\eta, \varepsilon, M)$ for all $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$.*
5. *The matrix elements $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ satisfy for all $(\varepsilon, M) \in \mathfrak{D}_0 \setminus \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ the bounds*
$$\left| L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M) \right| \le \mathrm{e}^{-\kappa |\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} |\eta|^N K_1, \qquad |\partial_\varepsilon L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)| \le \mathrm{e}^{-\kappa |\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} |\eta|^N K_1 |p_{\boldsymbol{\nu}}|^{c_0},$$
$$|\partial_\eta L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)| \le \mathrm{e}^{-\kappa |\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} N \, |\eta|^{N-1} K_1,$$
for all $(\varepsilon, M) \in \mathfrak{D}_0 \setminus \cup \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, where the union is taken over all the resonant pairs $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$, one has
$$\left| \partial_M L^E(\eta, \varepsilon, M) \right|_{\mathrm{op}} \le \sum_{\boldsymbol{\nu} \in \mathfrak{D}} \sum_{\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}} \sum_{\sigma,\sigma' = \pm} \left| \partial_{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}} L^E(\eta, \varepsilon, M) \right|_\kappa \le |\eta|^N K_1,$$
and, finally, one has
$$\left| u^{E\,\sigma}_{\boldsymbol{\nu}}(\eta, \varepsilon, M) \right| \le |\eta|^N K_1 \mathrm{e}^{-\kappa |\boldsymbol{\nu}|^{1/2}},$$
uniformly for $(\varepsilon, M) \in \mathfrak{D}_0$.

Remark 11. *In our analysis we choose $M \in B_\kappa$ because eventually we obtain $L \in B_\kappa$, but – as the bound on the M-derivative in item 5 of Proposition 1 suggests – we could also take M in a larger space, say B_∞ with norm $|M|_\infty = \sup_{\boldsymbol{\nu},\boldsymbol{\nu}' \in \mathfrak{D}} \sup_{\sigma,\sigma'=\pm} |M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}|$.*

Once we have proved Proposition 1, we solve the compatibility equation (3.5) for the extended counterterms $L^E(\varepsilon^{1/N}, \varepsilon, M)$, which are well defined provided we choose $\varepsilon < \varepsilon_0$, with $\varepsilon_0 = \eta_0^N$.

Proposition 2. *There exist C^1 functions $\varepsilon \to (\varepsilon, M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon))$ from $\mathfrak{E}_0 \setminus \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma) \to \mathfrak{D}_0$, with an appropriate choice of C_0 in Definition 9, such that the following holds.*

1. *$M(\varepsilon)$ verifies the equation*
$$M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon)), \tag{3.6}$$

and the bounds

$$\left|M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\right| \leq K_2\varepsilon \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho}, \qquad \left|\partial_\varepsilon M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\right| \leq K_2\left(1+\varepsilon p^{c_0}_{\boldsymbol{\nu}}(\varepsilon)\right)\mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho},$$

for a suitable constant K_2*.*

2. *The set* $\mathfrak{E}(2\gamma) := \{\varepsilon \in \mathfrak{E}_0 : (\varepsilon, M(\varepsilon)) \in \mathfrak{D}(2\gamma)\}$ *has large relative Lebesgue measure, namely* $\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\mathrm{meas}(\mathfrak{E}(2\gamma)\cap(0,\varepsilon)) = 1$*.*

3.4 Proof of Theorem 1

By item 1 in Proposition 1 for all $(\varepsilon, M) \in \mathfrak{D}(\gamma)$ we can find a matrix $L(\eta,\varepsilon,M)$ so that there exists a unique solution $u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M)$ of (3.3) and (3.4) for all $|\eta| \leq \eta_0$, for a suitable η_0, and for ε_0 small enough. By item 3 in Proposition 1 the matrix elements $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta,\varepsilon,M)$ and the solution $u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M)$ can be extended to C^1 functions – denoted by $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta,\varepsilon,M)$ and $u^{E\,\sigma}_{\boldsymbol{\nu}}(\eta,\varepsilon,M)$ – for all $(\varepsilon,M) \in \mathfrak{D}_0 \setminus \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ and for all $(\varepsilon,M) \in \mathfrak{D}_0$, respectively. Moreover $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta,\varepsilon,M) = L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta,\varepsilon,M)$ and $u^{E\,\sigma}_{\boldsymbol{\nu}}(\eta,\varepsilon,M) = u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M)$ for all $(\varepsilon,M) \in \mathfrak{D}(2\gamma)$.

Equation (3.3) coincides with our original (3.2) provided the compatibility equation (3.5) is satisfied. Now we fix $\varepsilon_0 < \eta_0^N$ so that $L^E(\varepsilon^{1/N},\varepsilon,M)$ and $u^{E\,\sigma}_{\boldsymbol{\nu}}(\varepsilon^{1/N},\varepsilon,M)$ are well defined for $|\varepsilon| < \varepsilon_0$. By item 1 in Proposition 2, there exists a matrix $M(\varepsilon)$ which satisfies the extended compatibility equation (3.6). Finally by item 2 in Proposition 2 the Cantor set $\mathfrak{E}(2\gamma)$ is well defined and of large relative measure.

For all $\varepsilon \in \mathfrak{E}(2\gamma)$ the pair $(\varepsilon, M(\varepsilon))$ is by definition in $\mathfrak{D}(2\gamma)$, so that by item 3 in Proposition 1 one has $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N},\varepsilon,M(\varepsilon)) = L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N},\varepsilon,M(\varepsilon))$ and $u^\sigma(\varepsilon^{1/N},\varepsilon,M(\varepsilon);x,t) = u^{E\,\sigma}(\varepsilon^{1/N},\varepsilon,M(\varepsilon);x,t)$, and hence $u^\sigma_{\boldsymbol{\nu}}(\varepsilon^{1/N},\varepsilon,M(\varepsilon))$ solves (3.3) for $\eta = \varepsilon^{1/N}$. So, by item 1 in Proposition 2, $M(\varepsilon)$ solves the true compatibility equation (3.5) for all $\varepsilon \in \mathfrak{E}(2\gamma)$. Then $u^\sigma(\varepsilon^{1/N},\varepsilon,M(\varepsilon);x,t)$ is a true nontrivial solution of (3.3) and (3.4) in $\mathfrak{E}(2\gamma)$. Then by setting $\mathfrak{E} = \mathfrak{E}(2\gamma)$ the result follows.

4 Tree expansion

4.1 Recursive equations

In this section we find a formal solution $u^\sigma_{\boldsymbol{\nu}}, L$ of (3.3) and (3.4) as a power series on η; the solution $u^\sigma_{\boldsymbol{\nu}}, L$ depends on the matrix M and it will be written in the form of a tree expansion.

We assume for $u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M)$ for all $\boldsymbol{\nu} \in \mathfrak{P}$ and for the matrix $L(\eta,\varepsilon,M)$ a formal series expansion in η:

$$u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M) = \sum_{k=N}^\infty \eta^k u^{(k)\sigma}_{\boldsymbol{\nu}}, \qquad L(\eta,\varepsilon,M) = \sum_{k=N}^\infty \eta^k L^{(k)}, \tag{4.1}$$

with the Ansatz that $L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = 0$ if either $\bar\chi_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\bar\chi_1(\delta_{\boldsymbol{\nu}'}(\varepsilon)) = 0$ or the pair $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$ is not resonant, so that $L = \widehat\chi_1 L\widehat\chi_1$. We set also $u^{(k)\sigma}_{\boldsymbol{\nu}} = 0$ for all $k \leq N$ and $\boldsymbol{\nu},\boldsymbol{\nu}' \in \mathfrak{P}$ same for $L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ for $\boldsymbol{\nu},\boldsymbol{\nu}' \in \mathfrak{O}$.

For $\boldsymbol{\nu} \in \mathfrak{Q}$ we set

$$u^\sigma_{\boldsymbol{\nu}}(\eta,\varepsilon,M) = u^{(0)\sigma}_{\boldsymbol{\nu}} + \sum_{k=N}^\infty \eta^k u^{(k)\sigma}_{\boldsymbol{\nu}}. \tag{4.2}$$

with $u^{(0)+}_{\boldsymbol{\nu}} = u^{(0)}_{\boldsymbol{\nu}}$ and $u^{(0)-}_{\boldsymbol{\nu}} = \overline{u^{(0)}_{\boldsymbol{\nu}}}$ (cf. item 2 in Hypothesis 2 for notations). Again we set $u^{(k)\sigma}_{\boldsymbol{\nu}} = 0$ for $0 < k < N$ and $\boldsymbol{\nu} \in \mathfrak{Q}$.

Inserting the series expansions (4.1) and (4.2) into (3.3) we obtain

$$\begin{cases} u_{\boldsymbol{\nu}}^{(k)\sigma} = \dfrac{f_{\boldsymbol{\nu}}^{(k-N)\sigma}}{\delta_{\boldsymbol{\nu}}(\varepsilon)}, & \boldsymbol{\nu} \in \mathfrak{R}, \\ u_{\boldsymbol{\nu}}^{(k)\sigma} = \displaystyle\sum_{\boldsymbol{\nu}' \in \mathfrak{Q},\, \sigma'=\pm} (J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} f_{\boldsymbol{\nu}'}^{(k)\sigma'}, & \boldsymbol{\nu} \in \mathfrak{Q}, \\ \left(\mathbb{D}(\varepsilon) + \widehat{M}\right) U^{(k)} = F^{(k-N)} + \displaystyle\sum_{r=N}^{k-N} L^{(r)} U^{(k-r)}. & \end{cases} \tag{4.3}$$

4.2 Multiscale analysis

Definition 11. ***(The scale functions).*** *Let χ be a non-increasing function $C^\infty(\mathbb{R}_+, [0,1])$, such that $\chi(x) = 0$ if $x \geq 2\gamma$ and $\chi(x) = 1$ if $x \leq \gamma$; moreover one has $|\partial_x \chi(x)| \leq \Gamma\gamma^{-1}$ for some positive constant Γ. Let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1}x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$.*

Recall that for each ε the matrix $A = \mathbb{D}(\varepsilon) + \widehat{M}$ is block diagonal with a diagonal part whose eigenvalues are larger than $\bar{\gamma} > \gamma$ and a list of $C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon) \times C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)$ blocks $A^{\boldsymbol{\nu}}$ containing small entries. In the following if $A^{\boldsymbol{\nu}}$ is invertible – i.e. if $x_{\boldsymbol{\nu}} \neq 0$ – we will denote the entries of $(A^{\boldsymbol{\nu}})^{-1}$ by $(A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}$ even though it may be possible that the whole matrix A is not invertible.

Definition 12. ***(Propagators).*** *For $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{O}$, we define the* propagators

$$(G_{i,h})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = \begin{cases} \chi_h(x_{\boldsymbol{\nu}}(\varepsilon))\, \bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon)) \bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon)) (A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}, & \textit{if } i = 1 \textit{ and } \chi_h(x_{\boldsymbol{\nu}}(\varepsilon)) \neq 0, \\ \bar{\chi}_0(\delta_{\boldsymbol{\nu}}(\varepsilon))\, \delta_{\boldsymbol{\nu}}^{-1}(\varepsilon), & \textit{if } i = 0,\ \boldsymbol{\nu} = \boldsymbol{\nu}',\ \sigma = \sigma' \textit{ and } h = -1, \\ 0, & \textit{otherwise.} \end{cases}$$

In terms of the propagators we obtain

$$A^{-1} = \sum_{i=0,1} \sum_{h=-1}^{\infty} G_{i,h}, \tag{4.4}$$

which provides the multiscale decomposition. Notice that if $(A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} \neq 0$ then $x_{\boldsymbol{\nu}}(\varepsilon) = x_{\boldsymbol{\nu}'}(\varepsilon)$ (see Remark 10), so that the matrices $G_{i,h}$ are indeed self-adjoint.

Remark 12. *Only the propagator $G_{1,h}$ can produce small divisors while the propagator $G_{0,-1}$ is diagonal and of order one. Hence, there exists a positive constant C such that we can bound the propagators as*

$$|G_{0,-1}|_\infty \leq C\gamma^{-1}, \qquad \left|(G_{1,h})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\right| \leq 2^h C \gamma^{-1} p_{\boldsymbol{\nu}}^{-\xi}(\varepsilon) \sqrt{p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)}, \tag{4.5}$$

where the condition $d_{\boldsymbol{\nu}}(\varepsilon) \leq 2C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)$ – cf. Remark 10 – and item 2 of Lemma 4 have been used.

We write $L^{(k)}$ in (4.1) as

$$L_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2} = \sum_{h=-1}^{\infty} \chi_h(x_{\boldsymbol{\nu}_1}(\varepsilon)) L_{h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2}, \tag{4.6}$$

for all resonant pairs $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\}$; we denote by $L_h^{(k)}$ the matrix with entries $L_{h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2}$. Finally we set

$$U^{(k)} = \sum_{i=0,1} \sum_{h=-1}^{\infty} U_{i,h}^{(k)}, \tag{4.7}$$

so that (4.3) gives

$$
\begin{cases}
u_{\boldsymbol{\nu}}^{(k)\sigma} = \displaystyle\sum_{\boldsymbol{\nu}'\in\mathfrak{Q}} (J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} f_{\boldsymbol{\nu}'}^{(k)\sigma'}, & \boldsymbol{\nu}\in\mathfrak{Q}, \\
u_{\boldsymbol{\nu}}^{(k)\sigma} = \dfrac{f_{\boldsymbol{\nu}}^{(k-N)\sigma}}{\delta_{\boldsymbol{\nu}}(\varepsilon)}, & \boldsymbol{\nu}\in\mathfrak{R}, \\
U_{i,h}^{(k)} = G_{i,h}F^{(k-N)} + \delta(i,1)\, G_{1,h} \displaystyle\sum_{h_1=-1}^{\infty}\sum_{r=N}^{k-N} L_h^{(r)} U_{1,h_1}^{(k-r)}, & i=0,1,\ h\geq -1,
\end{cases}
\tag{4.8}
$$

which are the recursive equations we want to study.

4.3 Diagrammatic rules

A connected graph $\mathcal{G}$ is a collection of points (vertices) and lines connecting all of them. We denote with $V(\mathcal{G})$ and $L(\mathcal{G})$ the set of nodes and the set of lines, respectively. A path between two nodes is the minimal subset of $L(\mathcal{G})$ connecting the two nodes. A graph is planar if it can be drawn in a plane without graph lines crossing.

Definition 13. ***(Trees).*** *A tree is a planar graph $\mathcal{G}$ containing no closed loops. One can consider a tree $\mathcal{G}$ with a single special node v_0: this introduces a natural partial ordering on the set of lines and nodes, and one can imagine that each line carries an arrow pointing toward the node v_0. We can add an extra (oriented) line ℓ_0 exiting the special node v_0; the added line ℓ_0 will be called the root line and the point it enters (which is not a node) will be called the root of the tree. In this way we obtain a rooted tree θ defined by $V(\theta) = V(\mathcal{G})$ and $L(\theta) = L(\mathcal{G}) \cup \ell_0$. A labelled tree is a rooted tree θ together with a label function defined on the sets $L(\theta)$ and $V(\theta)$.*

We shall call *equivalent* two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labelled trees, simply by considering equivalent two labelled trees if they can be transformed into each other in such a way that also the labels match.

Given two nodes $v, w \in V(\theta)$, we say that $v \prec w$ if w is on the path connecting v to the root line. We can identify a line with the nodes it connects; given a line $\ell = (w, v)$ we say that ℓ enters w and exits (or comes out of) v, and we write $\ell = \ell_v$. Given two comparable lines ℓ and ℓ_1, with $\ell_1 \prec \ell$, we denote with $\mathcal{P}(\ell_1, \ell)$ the path of lines connecting ℓ_1 to ℓ; by definition the two lines ℓ and ℓ_1 do not belong to $\mathcal{P}(\ell_1, \ell)$. We say that a node v is along the path $\mathcal{P}(\ell_1, \ell)$ if at least one line entering or exiting v belongs to the path. If $\mathcal{P}(\ell_1, \ell) = \emptyset$ there is only one node v along the path (such that ℓ_1 enters v and ℓ exits v).

Definition 14. ***(Lines and nodes).*** *We call* internal nodes *the nodes such that there is at least one line entering them; we call* internal lines *the lines exiting the internal nodes. We call* end-nodes *the nodes which have no entering line. We denote with $L(\theta)$, $V_0(\theta)$ and $E(\theta)$ the set of lines, internal nodes and end-nodes, respectively. Of course $V(\theta) = V_0(\theta) \cup E(\theta)$.*

We associate with the nodes (internal nodes and end-nodes) and lines of any tree θ some labels, according to the following rules.

Definition 15. ***(Diagrammatic rules).***

1. *For each node v there are $p_v \geq 0$ entering lines. If $p_v = 0$ then $v \in E(\theta)$, if $p_v > 0$ then either $p_v = 1$ or $p_v \geq N+1$ and $v \in V_0(\theta)$. If $L(v)$ is the set of lines entering v one has $p_v = |L(v)|$.*

2. *With each internal line $\ell \in L(\theta)$ one associates a label q, p or r. We say that ℓ is a p-line, a q-line or an r-line, respectively, and we call $L_q(\theta)$, $L_p(\theta)$ and $L_r(\theta)$ the set of internal lines $\ell \in L(\theta)$ which are q-lines, p-lines and r-lines, respectively. If $p_v = 1$ then the line ℓ exiting v and the line ℓ_1 entering v are both p-lines.*

3. *With each line $\ell \in L(\theta)$ one associates the* type *label $i_\ell = 0, 1$.*

4. *With each line $\ell \in L(\theta)$ except the root line ℓ_0 one associates a* sign *label $\sigma_\ell = \pm$.*

5. *With each internal line $\ell \in L(\theta)$ one associates the* momenta *$(\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell) \in \mathbb{Z}^{D+1} \times \mathbb{Z}^{D+1}$.*

6. *With each line $\ell \in L(\theta)$ exiting an end-node one associates the* momentum *$\boldsymbol{\nu}_\ell$.*

7. *With each line $\ell \in L(\theta)$ one associates the* scale *label $h_\ell \in \mathbb{N} \cup \{-1, 0\}$.*

8. *With each end-node $v \in E(\theta)$ one associates the* mode *label $\boldsymbol{\nu}_v \in \mathfrak{Q}$, the* order *label $k_v = 0$, and the* sign *label $\sigma_v = \pm$.*

9. *With each internal node $v \in V_0(\theta)$ one associates the* mode *label $m_v \in \mathbb{Z}^D$, the* order *label $k_v \in \mathbb{N}$, and the* sign *label $\sigma_v = \pm$.*

10. *For each internal node $v \in V_0(\theta)$ one defines r_v as the number of lines $\ell \in L(v)$ with $\sigma_\ell = \sigma_v$, and one sets $s_v = p_v - r_v$.*

11. *If a line $\ell \in L(\theta)$ is not a p-line one sets $i_\ell = 0$*

12. *If a line $\ell \in L(\theta)$ has $i_\ell = 0$, then $h_\ell = -1$.*

13. *Let $\ell \in L(\theta)$ be an internal line. If ℓ is a p-line with $i_\ell = 0$, then $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}'_\ell$. If ℓ is a p-line with $i_\ell = 1$, then $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell\}$ is a resonant pair. If ℓ is a q-line, then $\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell \in \mathfrak{Q}$. If ℓ is an r-line, then $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}'_\ell \in \mathfrak{R}$.*

14. *If ℓ exits an end-node v, then $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_v$.*

15. *If two p-lines ℓ and ℓ' have $i_\ell = i_{\ell'} = 1$ and are such that $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}, \boldsymbol{\nu}'_{\ell'}\}$ is a resonant set, then $|h_\ell - h_{\ell'}| \leq 1$.*

16. *If $\ell \in L(\theta)$ exits an end-node $v \in E(\theta)$, then one sets $\sigma_\ell = \sigma_v$.*

17. *If ℓ is the line exiting v and $\ell_1, \ldots, \ell_{p_v}$ are the lines entering v one has*

$$\boldsymbol{\nu}'_\ell = (0, m_v) + \sigma_v(\sigma_{\ell_1}\boldsymbol{\nu}_{\ell_1} + \ldots + \sigma_{\ell_{p_v}}\boldsymbol{\nu}_{\ell_{p_v}}) = (0, m_v) + \sigma_v \sum_{\ell' \in L(v)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'},$$

    which represents a conservation rule for the momenta.

18. *Given an internal node $v \in V_0(\theta)$, if $p_v = 1$ one has $k_v \geq N$, while if $p_v \geq N$ one has $k_v = p_v - 1$.*

19. *Given an internal node $v \in V_0(\theta)$, if $p_v = 1$, let ℓ_1 be the line entering v and ℓ be the line exiting v. One has $i_{\ell_1} = i_\ell = 1$ and $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair.*

20. *With each end-node $v \in E(\theta)$ one associates the* node factor *$\eta_v = u^{(0)\sigma_v}_{\boldsymbol{\nu}_v}$; cf. item 2 in Hypothesis 2 and (4.2) for notations.*

21. *With each internal node $v \in V_0(\theta)$ with $p_v > 1$ one associates the* node factor *$\eta_v = a^{\sigma_v}_{r_v, s_v, m_v}$, where $a^\sigma_{r,s,m}$ satisfies equation (1.11).*

22. *With each internal node $v \in V_0(\theta)$ with $p_v = 1$ one associates the* node factor $\eta_v = L^{(k_v)\sigma_v,\sigma_{\ell_1}}_{h_\ell,\boldsymbol{\nu}'_\ell,\boldsymbol{\nu}_{\ell_1}}$, *still to be defined (see Definition 25 below), where ℓ and ℓ_1 are the lines exiting and entering v, respectively.*

23. *One associates with each line $\ell \in L(\theta)$ a* line propagator $g_\ell \in \mathbb{C}$ *with the following rules. If ℓ is a p-line exiting the internal node v one sets $g_\ell := (G_{i_\ell,h_\ell})^{\sigma_\ell,\sigma_v}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell}$, if ℓ is an r-line one sets $g_\ell := 1/\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)$, if ℓ is a q-line exiting the internal node v one sets $g_\ell := (J^{-1})^{\sigma_\ell,\sigma_v}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell}$, if ℓ exits an end-node one sets $g_\ell = 1$.*

24. *One defines the* order *of the tree θ as*

$$k(\theta) := \sum_{v \in V(\theta)} k_v,$$

the momentum *of θ as the momentum $\boldsymbol{\nu}_\ell$ of the root line ℓ, and the* sign *of θ as the sign σ_{v_0} of the node v_0 which the root line exits.*

Definition 16. ***(The sets of trees $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ and Θ).*** *We call $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ the set of all the nonequivalent trees of order k, momentum $\boldsymbol{\nu}$ and sign σ, defined according to the diagrammatic rules of Definition 15. We call Θ the sets of trees belonging to $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ for some $k \geq 1$, $\sigma = \pm$ and $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$.*

4.4 Clusters and resonances

Definition 17. ***(Clusters).*** *Given a tree $\theta \in \Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ a* cluster *T on scale h is a connected maximal set of nodes and lines such that all the lines ℓ have a scale label $\leq h$ and at least one of them has scale h; we shall call $h_T = h$ the scale of the cluster. We shall denote by $V(T)$, $V_0(T)$ and $E(T)$ the set of nodes, internal nodes and the set of end-nodes, respectively, which are contained inside the cluster T, and with $L(T)$ the set of lines connecting them. Finally $k(T) = \sum_{v \in V(T)} k_v$ will be called the order of T.*

An inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. A cluster T can have an arbitrary number of lines entering it (*entering lines*), but only one or zero line coming out from it (*exiting line* or *root line* of the cluster); we shall denote the latter (when it exists) with ℓ_T. Notice that, by definition, $|V(T)| > 1$ and all the entering and exiting lines have $i_\ell = 1$.

Definition 18. ***(Resonances).*** *We call* resonance *on scale h a cluster T on scale $h_T = h$ such that*

1. *the cluster has only one entering line ℓ^1_T and one exiting line ℓ_T of scale $h_{\ell_T} \geq h + 2$,*
2. *one has that $\{\boldsymbol{\nu}'_{\ell_T}, \boldsymbol{\nu}_{\ell^1_T}\}$ is a resonant pair and $\min\{|\boldsymbol{\nu}_{\ell^1_T}|, |\boldsymbol{\nu}'_{\ell_T}|\} \geq 2^{(h-2)/\tau}$,*
3. *for all $\ell \in \mathcal{P}(\ell^1_T, \ell_T)$ with $i_\ell = 1$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell^1_T}\}$ is not resonant,*
4. *for all $\ell \in L(T) \setminus \mathcal{P}(\ell^1_T, \ell_T)$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell^1_T}\}$ is not resonant.*

The line ℓ_T of a resonance will be called the root line of the resonance.

Definition 19. ***(The sets of trees $\mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $\mathcal{R}$).*** *For $k \geq N$, $h \geq 1$ and a resonant pair $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ such that $\min\{|\boldsymbol{\nu}|, |\boldsymbol{\nu}'|\} \geq 2^{(h-2)/\tau}$, we define $\mathcal{R}^{(k)\sigma\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ as the set of trees with the following differences with respect to $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$.*

1. *There is a single end-node, called e, with node factor $\eta_e = 1$ (but no label no labels $\boldsymbol{\nu}_e$ nor σ_e).*

2. *The line ℓ_e exiting e is a p-line. We associate with ℓ_e the labels $\boldsymbol{\nu}_{\ell_e} = \boldsymbol{\nu}'$, $\sigma_\ell = \sigma'$, and $i_{\ell_e} = 1$ (but no labels $\boldsymbol{\nu}'_\ell$ nor h_ℓ), and the corresponding line propagator is $g_{\ell_e} = \bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon))$.*
3. *The root line ℓ_0 is a p-line. We associate with ℓ_0 the labels $i_{\ell_0} = 1$ and $\boldsymbol{\nu}'_{\ell_0} = \boldsymbol{\nu}$ (but no labels $\boldsymbol{\nu}_{\ell_0}$ nor h_{ℓ_0}), and the corresponding line propagator is $g_{\ell_0} = \bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))$. Let v_0 be the node which the line ℓ_0 exits: we set $\sigma_{v_0} = \sigma$.*
4. *One has $\max_{\ell \in L(\theta) \setminus \{\ell_0, \ell_e\}} h_\ell = h$.*
5. *If $\ell \in \mathcal{P}(\ell_e, \ell_0)$ is such that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is resonant, then $i_\ell = 0$.*
6. *For $\ell \notin \mathcal{P}(\ell_e, \ell_0)$ one has that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair.*

We call $\mathcal{R}$ the sets of trees belonging to $\mathcal{R}^{(k)\sigma\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \geq 1$, $h \geq 1$, $\sigma, \sigma' = \pm$, and $\boldsymbol{\nu}, \boldsymbol{\nu} \in \mathfrak{O}$ such that $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ is resonant and $\min\{|\boldsymbol{\nu}|, |\boldsymbol{\nu}'|\} \geq 2^{(h-2)/\tau}$.

Definition 20. ***(Clusters for trees in $\mathcal{R}$).*** *Given a tree $\theta \in \mathcal{R}$, a cluster T on scale $h_T \leq h$ is a connected maximal set of nodes $v \in V(\theta)$ and lines $\ell \in L(\theta) \setminus \{\ell_0, \ell_e\}$ such that all the lines ℓ have a scale label $\leq h_T$ and at least one of them has scale h_T.*

Note that if $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, then for any cluster T in θ one necessarily has $h_T \leq h$.

Definition 21. ***(Resonances for trees in $\mathcal{R}$).*** *Given a tree $\theta \in \mathcal{R}$, a cluster T is a resonance if the four items of Definition 18 are satisfied.*

Remark 13. *There is a one-to-one correspondence between resonances T of order k and scale h with $\boldsymbol{\nu}_{\ell_T^1} = \boldsymbol{\nu}'$, $\boldsymbol{\nu}'_{\ell_T} = \boldsymbol{\nu}$, $\sigma_{v_0} = \sigma$, $\sigma_{\ell_T^1} = \sigma'$ (here v_0 is the node which ℓ_T exits) and trees $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$; cf. [18], Section 3.4 and Figure 7.*

Definition 22. ***(The sets of renormalised trees $\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$, $\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, Θ_R and $\mathcal{R}_R$).*** *We define the set of* renormalised trees $\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ *and* $\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ *as the set of trees defined as $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ and $\mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, respectively, but with no resonances and no nodes v with $p_v = 1$. Analogously we define the sets Θ_R and $\mathcal{R}_R$.*

In the following it will turn out to be convenient to introduce also the following set of trees.

Definition 23. ***(The set of renormalised trees $\mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $\mathcal{S}_R$).*** *For $k \geq N$, $h \geq 1$ and $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{O}$ such that $|\boldsymbol{\nu}'| \geq 2^{(h-2)/\tau}$ we define the set of renormalised trees $\mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ as the set of trees with the following differences with respect to $\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ (see Definition 19).*
Items 1 *and* 2 *are unchanged.*

3′ One assigns to the line ℓ_0 the further label $h_{\ell_0} \leq h$, and requires $|\boldsymbol{\nu}| \geq 2^{(h_{\ell_0}-2)/\tau}$.

4′ One has $\max_{\ell \in L(\theta) \setminus \{\ell_e\}} h_\ell = h$

Items 5 *and* 6 *are unchanged.*
The set $\mathcal{S}_R$ is defined analogously as $\mathcal{R}_R$.

Remark 14. *Note that if $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ then $\mathrm{Val}(\theta) = \mathrm{Val}(\theta')$ with $\theta' \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ such that $h_{\ell_0} = h - 1$. Thus, it is enough to study the set $\mathcal{S}_R$ in order to obtain bounds for trees in $\mathcal{R}_R$.*

Definition 24. ***(Tree values).*** *For any tree or renormalised tree θ call*

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell \Big) \Big(\prod_{v \in V(\theta)} \eta_v \Big)$$

the value *of the tree θ. To make explicit the dependence of the tree value on ε and M, sometimes we shall write* $\mathrm{Val}(\theta) = \mathrm{Val}(\theta; \varepsilon, M)$.

Definition 25. ***(Counterterms).*** *We define the node factors* $L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ *(cf. item 21 in Definition 15) by setting*

$$L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'} = \sum_{h'<h-1} \sum_{\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h',\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta), \qquad \sigma,\sigma' = \pm, \tag{4.9}$$

for all $k \geq N$*, all* $h \geq 1$*, and all resonant pairs* $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$*. The counterterms* L *are then expressed in terms of (4.9) through (4.1) and (4.6).*

Lemma 5. *For any tree* $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ *there exists a tree* $\theta' \in \mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$ *such that* $\mathrm{Val}(\theta) = \mathrm{Val}(\theta')$*.*

Proof. Given a tree $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, consider the path $\mathcal{P} = \mathcal{P}(\ell_e,\ell_0)$, and set $\mathcal{P} = \{\ell_1,\ldots,\ell_N\}$, with $\ell_0 \succ \ell_1 \succ \ldots \succ \ell_N \succ \ell_{N+1} = \ell_e$ (if $\mathcal{P} = \emptyset$, set $N = 0$ in the forthcoming discussion). For $k = 0,\ldots,N$, denote by v_k the node which the line ℓ_k exits and by $L_0(v_k)$ the set $L(v_k)\setminus\{\ell_{k+1}\}$ (cf. item 1 in Definition 15).

We construct a tree $\theta' \in \mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$ in the following way.

1. We shift the sign labels down the path $\mathcal{P}$ and change their sign, so that $\sigma_{\ell_k} \to -\sigma_{v_k}$ and $\sigma_{v_k} \to -\sigma_{\ell_{k+1}}$ for $k = 0,\ldots,N$. In particular ℓ_0 acquires the label $-\sigma_{v_0}$, while ℓ_e loses its label σ_{ℓ_e} (which with the opposite sign becomes associated with the node v_N).
2. The end-node e becomes the root, and the root line becomes the end-node e. In particular the line ℓ_e becomes the root line, and the line ℓ_0 becomes the entering line, so that the arrows of all the lines $\ell \in \mathcal{P}$ are reverted, while the ordering of all the lines and nodes outside $\mathcal{P}$ is not changed.
3. For all the lines $\ell \in \mathcal{P}$ we exchange the labels $\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell$, so that $\boldsymbol{\nu}_{\ell_k} \to \boldsymbol{\nu}'_{\ell_k}$ and $\boldsymbol{\nu}'_{\ell_k} \to \boldsymbol{\nu}_{\ell_k}$ for $k = 1,\ldots,N$, and we set $\boldsymbol{\nu}'_{\ell_e} = \boldsymbol{\nu}'$ and $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}$.
4. For all $k = 0,\ldots,N$ we replace $m_{v_k} \to -\sigma_{v_k}\sigma_{\ell_{k+1}} m_{v_k}$.

By construction, the tree θ' belongs to $\mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$, and all line propagators and node factors of the lines and nodes, respectively, which do not belong to $\mathcal{P}$ remain the same.

Moreover, the line propagator of each $\ell_k \in \mathcal{P}$ in θ' is $(G_{i_{\ell_k},h_{\ell_k}})^{-\sigma_{v_k},-\sigma_{\ell_k}}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_{\ell_k}} = (G_{i_{\ell_k},h_{\ell_k}})^{\sigma_{\ell_k},\sigma_{v_k}}_{\boldsymbol{\nu}'_{\ell_k},\boldsymbol{\nu}_\ell}$, hence it does not change with respect with the line propagator of the corresponding line in θ. For each node v_k, the conservation law

$$\boldsymbol{\nu}_{\ell_{k+1}} = (0, -\sigma_{v_k}\sigma_{\ell_{k+1}} m_{v_k}) - \sigma_{\ell_{k+1}}\Big(-\sigma_{v_k}\boldsymbol{\nu}'_{\ell_k} + \sum_{\ell'\in L_0(v_k)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'}\Big) \tag{4.10}$$

is assured by the conservation law (cf. item 17 in Definition 15)

$$\boldsymbol{\nu}'_{\ell_k} = (0, m_{v_k}) + \sigma_{v_k}\Big(\sigma_{\ell_{k+1}}\boldsymbol{\nu}_{\ell_{k+1}} + \sum_{\ell'\in L_0(v_k)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'}\Big) \tag{4.11}$$

for the corresponding node v_k in θ: simply multiply (4.11) times $\sigma_{v_k}\sigma_{\ell_{k+1}}$ in order to obtain (4.10).

Finally we want to show that the product of the combinatorial factors times the node factors of the nodes $v_0,\ldots,v_N$ do not change. Take a node $v = v_k$, for $k = 0,\ldots,N$, and call r'_v and s'_v the number of lines $\ell' \in L_0(v)$ with $\sigma_{\ell'} = \sigma_v$ and $\sigma_{\ell'} = -\sigma_v$, respectively. Set $\sigma_v = \sigma$ and $\sigma_{\ell_{k+1}} = \sigma'$.

Consider first the case $\sigma' = \sigma$. In that case in θ one has $r_v = r'_v + 1$ and $s_v = s'_v$, and the combinatorial factor contains a factor r_v because there are r_v lines ℓ entering v with $\sigma_\ell = \sigma$. In θ' one has $\sigma_v \to -\sigma$, $r_v \to s'_v + 1$, $s_v \to r'_v$ and $m_v \to -m_v$ (because $\sigma\sigma' = 1$). Moreover the corresponding combinatorial factor contains a factor $(s_v + 1)$ because there are $s_v + 1$ lines ℓ entering v with $\sigma_\ell = -\sigma$.

Therefore, taking into account also the combinatorics, the node factor associated with the node v in θ is $(s_v+1)a^{-\sigma}_{s_v+1,r_v-1,-m_v} = r_v\, a^{\sigma}_{r_v,s_v,m_v}$, i.e. the same as in θ, by the condition (1.11).

Now, we pass to the case $\sigma=-\sigma'$. In that case in θ one has $r_v=r_v'$, $s_v=s_v'+1$. In θ' one has the same values for r_v, s_v and σ_v, so that, by using also that $-\sigma\sigma' m_v = m_v$ in such a case, the node factors $a^{\sigma_v}_{r_v,s_v,m_v}$ do not change. Of course the combinatorial factors do not change either.

In conclusion, one has $\mathrm{Val}(\theta)=\mathrm{Val}(\theta')$, which yields the assertion. ■

Remark 15. *By Lemma 5 we have that the matrix $L^{(k)}_h$ is self-adjoint, and the Definition 25 together with (4.6) implies that we can write*

$$L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = \sum_{h=-1}^{\infty} C_h(x_{\boldsymbol{\nu}}(\varepsilon)) \sum_{\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta), \qquad C_h(x) = \sum_{h'=h+2}^{\infty} \chi_h(x), \qquad \sigma=\pm,$$

for all $k\ge N$, all $h\ge 1$, and all resonant pairs $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$. By construction $x_{\boldsymbol{\nu}}(\varepsilon)=x_{\boldsymbol{\nu}'}(\varepsilon)$ whenever $L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}\neq 0$, so that also $L^{(k)}$ is self-adjoint. Finally we have that $L^{(k)}=\widehat{\chi}_1 L^{(k)}\widehat{\chi}_1$ (cf. the definition of the line propagators g_{ℓ_0} and g_{ℓ_e} for trees $\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ in Definition 19).

Lemma 6. *One has*

$$u^{(k)\sigma}_{\boldsymbol{\nu}} = \sum_{\theta\in\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}} \mathrm{Val}(\theta), \qquad \sigma=\pm, \tag{4.12}$$

for all $k\ge 1$ and all $\boldsymbol{\nu}\in\mathbb{Z}^{D+1}$.

Proof. For any given counterterm L, the coefficients $u^{(k)\sigma}_{\boldsymbol{\nu}}$ can be written as sums over tree values

$$u^{(k)\sigma}_{\boldsymbol{\nu}} = \sum_{\theta\in\Theta^{(k)\sigma}_{\boldsymbol{\nu}}} \mathrm{Val}(\theta).$$

This can be easily proved by induction, using the diagrammatic rules and definitions given in this section; we refer to Lemma 3.6 of [18] for details. Then, defining the counterterms according to Definition 25, all contributions arising from trees belonging to the set $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ but not to the set $\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ cancel out exactly – see Lemma 3.13 of [18] for further details – and hence the assertion follows. ■

5 Bryuno lemma and bounds

Given a tree $\theta\in\Theta_R$, call $\mathfrak{S}(\theta,\gamma)$ the set of $(\varepsilon,M)\in\mathfrak{D}_0$ such that for all $\ell\in L_p(\theta)$ with $i_\ell=1$ one has

$$\begin{cases} 2^{-h_\ell-1}\gamma \le |x_{\boldsymbol{\nu}_\ell}(\varepsilon)| \le 2^{-h_\ell+1}\gamma, & h_\ell\neq -1,\\ |x_{\boldsymbol{\nu}_\ell}(\varepsilon)|\ge\gamma, & h_\ell=-1,\end{cases} \tag{5.1}$$

and for all $\ell\in L_p(\theta)$ one has

$$\begin{cases} |\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)|\le\bar\gamma, \quad |\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)|\le\bar\gamma, & i_\ell=1,\\ \bar\gamma\le|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)|, & i_\ell=0.\end{cases} \tag{5.2}$$

Define also $\mathfrak{D}(\theta,\gamma)\subset\mathfrak{D}_0$ as the set of $(\varepsilon,M)\in\mathfrak{D}_0$ such that for all $\ell\in L_p(\theta)$ with $i_\ell=0$ one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)\pm\bar\gamma|\ge\gamma/|\boldsymbol{\nu}_\ell|^{\tau_1}$, while for all $\ell\in L_p(\theta)$ with $i_\ell=1$ one has

$$x_{\boldsymbol{\nu}_\ell}(\varepsilon)\ge\frac{\gamma}{p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon)}, \qquad |\delta_{\boldsymbol{\nu}}(\varepsilon)\pm\bar\gamma|\ge\frac{\gamma}{|\boldsymbol{\nu}|^{\tau_1}}\ \forall\boldsymbol{\nu}\in\mathcal{C}_{\boldsymbol{\nu}_\ell}\cup\mathcal{C}_{\boldsymbol{\nu}'_\ell}, \tag{5.3}$$

for some $\tau, \tau_1 > 0$. Note that the second condition in (5.3) does not depend on M.

Analogously, given a tree $\theta \in \mathcal{S}_R$, we call $\widetilde{\mathfrak{S}}(\theta,\gamma)$ the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that (5.1) holds for all $\ell \in L_p(\theta) \setminus \{\ell_e, \ell_0\}$ with $i_\ell = 1$ and (5.2) holds for all $\ell \in L_p(\theta)$, and we call $\widetilde{\mathfrak{D}}(\theta,\gamma)$ as the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that (5.3) holds for all $\ell \in L_p(\theta) \setminus \{\ell_e, \ell_0\}$ with $i_\ell = 1$, while for all $\ell \in L_p(\theta)$ with $i_\ell = 0$ one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon) \pm \bar{\gamma}| \geq \gamma/|\boldsymbol{\nu}_\ell|^{\tau_1}$.

Remark 16. *If $(\varepsilon, M) \in \mathfrak{S}(\theta,\gamma)$ then $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, while $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma)$ means that we can use the bounds (5.3) to estimate $\mathrm{Val}(\theta;\varepsilon,M)$. Analogous considerations hold for trees $\theta \in \mathcal{S}_R$.*

Remark 17. *If for some ε one has $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$ and for two comparable lines $\ell, \ell' \in L(\theta)$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}\}$ is resonant, then all the set $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}, \boldsymbol{\nu}'_{\ell'}\}$ is resonant. This motivates the condition in item 15 in Definition 15.*

Remark 18. *If $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ is such that $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, then $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \Delta_j(\varepsilon)$ for some j, so that $p_{\boldsymbol{\nu}}(\varepsilon) = p_{\boldsymbol{\nu}'}(\varepsilon)$ and $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \leq C_1 C_2 p^{\alpha+\beta}_{\boldsymbol{\nu}}(\varepsilon) \leq C_1 C_2 p^{2\alpha}_{\boldsymbol{\nu}}(\varepsilon)$. Moreover $p_{\boldsymbol{\nu}}(\varepsilon) \leq |\boldsymbol{\nu}|, |\boldsymbol{\nu}'| \leq 2 p_{\boldsymbol{\nu}}(\varepsilon)$. Such properties follow from Hypothesis 3 – cf. also Remark 6.*

Definition 26. ***(The quantity $\boldsymbol{N_h(\theta)}$).*** *Define $N_h(\theta)$ as the set of lines $\ell \in L(\theta)$ with $i_\ell = 1$ and scale $h_\ell \geq h$.*

Definition 27. ***(The quantity $\boldsymbol{K(\theta)}$).*** *Define*

$$K(\theta) = k(\theta) + \sum_{v \in V_0(\theta)} |m_v| + \sum_{\ell \in L_q(\theta)} |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell| + \sum_{v \in E(\theta)} |\boldsymbol{\nu}_v|,$$

where $k(\theta)$ is the order of θ.

Lemma 7. *There exists a constant B such the following holds.*

1. *For all $\theta \in \Theta_R$ and all lines $\ell \in L(\theta)$ one has $|\boldsymbol{\nu}_\ell| \leq B(K(\theta))^{1+4\alpha}$.*
2. *If $\theta \in \mathcal{S}_R$, for all lines $\ell \in L(\theta) \setminus (\mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0, \ell_e\})$ one has $|\boldsymbol{\nu}_\ell| \leq B(K(\theta))^{1+4\alpha}$, while for all lines $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ one has $|\boldsymbol{\nu}'_\ell| \leq B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha}$.*
3. *Given a tree θ let $\ell, \ell' \in L(\theta)$ be two comparable lines, with $\ell \prec \ell'$, such that $i_\ell = i_{\ell'} = 1$ and $i_{\ell''} = 0$ for all the lines $\ell'' \in \mathcal{P}(\ell, \ell')$. If $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \geq BK(\theta)^{1+4\alpha}$, then one has $\mathrm{Val}(\theta) = 0$ for all ε.*
4. *If $\theta \in \mathcal{S}_R$, $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ and, moreover, $i_{\ell'} = 0$ for all lines $\ell' \in \mathcal{P}(\ell_e, \ell)$, then $|\boldsymbol{\nu}'_\ell| \leq |\boldsymbol{\nu}_{\ell_e}| + B(K(\theta))^{1+4\alpha}$.*

Proof. Let us consider first trees $\theta \in \Theta_R$. The proof is by induction on the order of the tree $k = k(\theta)$. For $k = 1$ the bound is trivial. If the root line ℓ_0 is either a q-line or an r-line or a p-line with $i_{\ell_0} = 0$, again the bound follows trivially from the inductive bound. If ℓ_0 is a p-line with $i_{\ell_0} = 1$, call v_0 the node such that $\ell_0 = \ell_{v_0}$ and $\theta_1, \ldots, \theta_s$ the subtrees with root in v_0. By the inductive hypothesis and Hypothesis 3 one obtains, for a suitable constant C and taking B large enough, $|\boldsymbol{\nu}_\ell| \leq |m_{v_0}| + B\left(K(\theta) - 1 - |m_{v_0}|\right)^{1+4\alpha} + C\left(|m_{v_0}| + B(K(\theta) - 1 - |m_{v_0}|)\right)^{2\alpha(1+4\alpha)} \leq B(K(\theta))^{1+4\alpha}$, which proves the assertion for Θ_R in item 1.

As a byproduct also the bound for $\mathcal{S}_R$ is obtained, as far as lines $\ell \notin \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0, \ell_e\}$ are concerned. The bound $|\boldsymbol{\nu}'_\ell| \leq B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha}$ for the lines $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ can be proved similarly by induction. Thus, also item 2 is proved.

Given two comparable lines ℓ, ℓ' such that $i_{\ell''} = 0$ for all lines $\ell'' \in \mathcal{P}(\ell, \ell')$, then by momentum conservation one has $\min\{|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}|, |\boldsymbol{\nu}'_\ell + \boldsymbol{\nu}_{\ell'}|\} \leq B(K(\theta))^{1+4\alpha}$ in case (I) and $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \leq B(K(\theta))^{1+4\alpha}$ in case (II). This proves the bounds in item 3 in case (II) and in item 4 for both cases (I) and (II).

In case (I), if $i_\ell = i_{\ell'} = 1$ and $\max\{|\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)|, |\delta_{\boldsymbol{\nu}_{\ell'}}(\varepsilon)|\} < 1/2$, then $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \le |\boldsymbol{\nu}'_\ell + \boldsymbol{\nu}_{\ell'}|$ by item 5 in Hypothesis 1. On the other hand if $i_\ell = i_{\ell'} = 1$ and $\max\{|\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)|, |\delta_{\boldsymbol{\nu}_{\ell'}}(\varepsilon)|\} \ge 1/2$, one has $\mathrm{Val}(\theta; \varepsilon, M) = 0$. Hence item 3 follows also in case (I). ■

Lemma 8. *Given a tree $\theta \in \Theta_R$ such that $\mathfrak{D}(\theta,\gamma) \cap \mathfrak{S}(\theta,\gamma) \neq \emptyset$, for all $h \ge 1$ one has*

$$N_h(\theta) \le \max\{0, c\, K(\theta) 2^{(2-h)\beta/2\tau} - 1\},$$

where c is a suitable constant.

Proof. Define $E_h := c^{-1} 2^{(h-2)\beta/2\tau}$. So, we have to prove that $N_h(\theta) \le \max\{0, K(\theta)E_h^{-1} - 1\}$.

If a line ℓ is on scale $h \ge 0$ then $\gamma/p^\tau_{\boldsymbol{\nu}_\ell}(\varepsilon) < x_{\boldsymbol{\nu}_\ell}(\varepsilon) \le 2^{-h+1}\gamma$ by (5.1) and (5.3). Hence $B(K(\theta))^2 \ge B(K(\theta))^{1+4\alpha} \ge |\boldsymbol{\nu}_\ell| \ge p_{\boldsymbol{\nu}_\ell}(\varepsilon) > 2^{(h-1)/\tau}$, by Lemma 7, so that $K(\theta)E_h^{-1} \ge cB^{-1/2}2^{(h-1)/2\tau}2^{(2-h)\beta/2\tau} \ge 2$ for c suitably large. Therefore if a tree θ contains a line ℓ on scale h one has $\max\{0, K(\theta)E_h^{-1} - 1\} = K(\theta)E_h^{-1} - 1 \ge 1$.

The bound $N_h(\theta) \le \max\{0, K(\theta)E_h^{-1} - 1\}$ will be proved by induction on the order of the tree. Let ℓ_0 be the root line of θ and call $\theta_1, \ldots, \theta_m$ the subtrees of θ whose root lines $\ell_1, \ldots, \ell_m$ are the lines on scale $h_{\ell_i} \ge h-1$ and $i_{\ell_i} = 1$ which are the closest to ℓ_0.

If $h_{\ell_0} < h$ we can write $N_h(\theta) = N_h(\theta_1) + \ldots + N_h(\theta_m)$, and the bound follows by induction. If $h_{\ell_0} \ge h$ then $\ell_1, \ldots, \ell_m$ are the entering lines of a cluster T with exiting line ℓ_0; in that case we have $N_h(\theta) = 1 + N_h(\theta_1) + \ldots + N_h(\theta_m)$. Again the bound follows by induction for $m = 0$ and $m \ge 2$. The case $m = 1$ can be dealt with as follows.

If $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair, then either there exists a line $\ell \in \mathcal{P}(\ell_1, \ell_0)$ with $i_\ell = 1$ such that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair or there must be a line $\ell \in L(T) \setminus \mathcal{P}(\ell_1, \ell_0)$ with $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ a resonant pair. In fact, the first case is not possible: indeed, also $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}'_\ell\}$ would be resonant (cf. Remark 17), so that $|h_\ell - h_{\ell_0}| \le 1$ (cf. item 15 in Definition 15), and hence the contradiction $h - 2 \ge h_\ell \ge h_{\ell_0} - 1 \ge h - 1$ would follow. In the second case, one has $|\boldsymbol{\nu}'_\ell| \ge p_{\boldsymbol{\nu}_{\ell_1}}(\varepsilon) > 2^{(h-2)/\tau}$, hence if θ' is the subtree with root line ℓ, then one has $K(\theta) - K(\theta_1) > K(\theta') > 2E_h$, and the bound follows once more by the inductive hypothesis.

If $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}_{\ell_1}\}$ is not a resonant pair, call $\bar\ell$ the line along the path $\mathcal{P}(\ell_1, \ell_0) \cup \{\ell_1\}$ with $i_{\bar\ell} = 1$ closest to ℓ_0. Since $i_{\bar\ell} = 1$ and by hypothesis $h_{\bar\ell} < h-1$ then $\{\boldsymbol{\nu}_{\bar\ell}, \boldsymbol{\nu}_{\ell_0}\}$ is not a resonant pair (see item 15 in Definition 15). Call $\tilde T$ the set of nodes and lines preceding ℓ_0 and following $\bar\ell$, and define $K(T) = K(\theta) - K(\theta_1)$ and $K(\tilde T) = K(\theta) - K(\bar\theta)$, where $\bar\theta$ is the tree with root line $\bar\ell$. Set also $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_{\bar\ell}$ and $\boldsymbol{\nu}_0 = \boldsymbol{\nu}'_{\ell_0}$. One has $2|\bar{\boldsymbol{\nu}} - \boldsymbol{\nu}_0| \ge C_2(p_{\bar{\boldsymbol{\nu}}}(\varepsilon) + p_{\boldsymbol{\nu}_0}(\varepsilon))^\beta \ge C_2 p^\beta_{\boldsymbol{\nu}_0}(\varepsilon)$ (see Remark 6), so that by Lemma 7 one finds $B(K(\theta) - K(\theta_1))^2 \ge B(K(\tilde T))^2 \ge |\bar{\boldsymbol{\nu}} - \boldsymbol{\nu}_0| \ge C_2\, p^\beta_{\boldsymbol{\nu}_0}(\varepsilon)/2 \ge C_2 2^{(h-1)\beta/\tau}/2$. Hence $(K(\theta) - K(\theta_1))E_h^{-1} \ge K(T)E_h^{-1} \ge K(\tilde T)E_h^{-1} \ge 2$, provided c is large enough. This proves the bound. ■

Lemma 9. *There exists positive constants ξ_0 and D_0 such that, if $\xi > \xi_0$ in Definition 8, then for all trees $\theta \in \Theta_R$ and for all $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma) \cap \mathfrak{S}(\theta,\gamma)$ one has*

$$|\mathrm{Val}(\theta)| \le D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p^{-(\xi - \xi_0)}_{\boldsymbol{\nu}_\ell}(\varepsilon), \tag{5.4a}$$

$$|\partial_\varepsilon \mathrm{Val}(\theta)| \le D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p^{-(\xi - \xi_0)}_{\boldsymbol{\nu}_\ell}(\varepsilon), \tag{5.4b}$$

$$\sum_{\boldsymbol{\nu} \in \mathfrak{D}} \sum_{\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}} \sum_{\sigma,\sigma' = \pm} \left| \partial_{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta) \right| \le D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p^{-(\xi - \xi_0)}_{\boldsymbol{\nu}_\ell}(\varepsilon). \tag{5.4c}$$

Proof. The propagators are bounded according to (4.5), so that for all trees $\theta \in \Theta^{(k)}_{R,\boldsymbol{\nu}}$ one has

$$|\mathrm{Val}(\theta)| \le C^k \Big(\prod_{v \in V_0(\theta)} \mathrm{e}^{-A_2 |m_v|} \Big) \Big(\prod_{\ell \in L_q(\theta)} \mathrm{e}^{-\lambda_0 |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell|} \Big) \times \\ \times \Big(\prod_{v \in E(\theta)} \mathrm{e}^{-\lambda_0 |\boldsymbol{\nu}_v|} \Big) 2^{k h_0} \Big(\prod_{h=h_0+1}^{\infty} 2^{h N_h(\theta)} \Big) \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p^{-\xi}_{\boldsymbol{\nu}_\ell}(\varepsilon)\, p^{a_0}_{\boldsymbol{\nu}_\ell}(\varepsilon),$$

for arbitrary h_0 and for suitable constants C and a_0. For $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma) \cap \mathfrak{S}(\theta,\gamma)$ one can bound $N_h(\theta)$ through Lemma 8. Therefore, by choosing h_0 large enough the bound (5.4a) follows, provided $\xi - a_0 > 0$ and κ is suitably chosen.

When bounding $\partial_\varepsilon \mathrm{Val}(\theta)$, one has to consider derivatives of the line propagators, i.e. $\partial_\varepsilon g_\ell$. If ℓ is an r-line then $|\partial_\varepsilon g_\ell|$ is bounded proportionally to $|\boldsymbol{\nu}_\ell|^{c_0}$, whereas if ℓ is a p-line, then the derivative produces factors which admit bounds of the form

$$C p^{a_1}_{\boldsymbol{\nu}_\ell}(\varepsilon)\, 2^{2h_\ell} p^{c_0}_{\boldsymbol{\nu}_\ell}(\varepsilon) p^{-\xi}_{\boldsymbol{\nu}_\ell}(\varepsilon), \tag{5.5}$$

for suitable constants C and a_1; see the proof of Lemma 4.2 in [18] for details (and use item 3 in Hypothesis 1).

The extra factor 2^{h_ℓ} can be taken into account by bounding the product of line propagators with

$$2^{2h_0 k} \prod_{h=h_0+1}^{\infty} 2^{2h N_h(\theta)}.$$

One can bound $|\boldsymbol{\nu}_\ell| \le B(K(\theta))^2$, and use part of the exponential decaying factors $\mathrm{e}^{-A_2|m_v|}$, $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell|}$, and $\mathrm{e}^{-\lambda_0 |\boldsymbol{\nu}_v|}$, to control the contribution $\sum_{v \in V_0(\theta)} |m_v| + \sum_{\ell \in L_q(\theta)} |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell| + \sum_{v \in E(\theta)} |\boldsymbol{\nu}_v|$ to $K(\theta)$ (cf. Definition 27). Then, if ξ is large enough, so that $\xi - a_1 > 0$ for all possible values of a_1 in (5.5), the bound (5.4b) follows.

Also the bound (5.4c) can be discussed in the same way. We refer again to [18] for the details. ■

Remark 19. *Note that for $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma)$ the singularities of the functions $\bar\chi_1$ are avoided, so that $\partial_\varepsilon \bar\chi_1(\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)) = 0$ for all $\ell \in L(\theta)$. Note also that the bound (5.4c) is not really needed in the following.*

Lemma 10. *There are two positive constants B_2 and B_3 such that the following holds.*

1. *Given a tree $\theta \in \mathcal{S}_R$ such that $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, if $K(\theta) \le B_2 p^{\beta/2}_{\boldsymbol{\nu}_{\ell_e}}(\varepsilon)$ then for all lines $\ell \in \mathcal{P}(\ell_e,\ell_0)$ one has $i_\ell = 0$. Moreover for all such lines ℓ, if $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_e}\}$ is not a resonant pair, then one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| \ge 1/2$.*
2. *Given a tree $\theta \in \mathcal{R}_R$ such that $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, one has $|\boldsymbol{\nu}'_{\ell_0} - \boldsymbol{\nu}_{\ell_e}| \le B_3 (K(\theta))^{1/\rho}$, with ρ depending on α and β.*

Proof. Suppose that $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $\mathcal{P}(\ell_e,\ell_0)$ contains lines ℓ with $i_\ell = 1$ and consequently with $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ not resonant (cf. Definition 23). Let $\bar\ell$ be the one closest to ℓ_e; thus, one has $|\boldsymbol{\nu}'_{\bar\ell} - \boldsymbol{\nu}'| \ge C_3 (|\boldsymbol{\nu}'_{\bar\ell}| + |\boldsymbol{\nu}'|)^\beta \ge C_3 p^\beta_{\boldsymbol{\nu}'}(\varepsilon) = C_3 p^\beta_{\boldsymbol{\nu}}(\varepsilon)$, so that we can apply item 3 in Lemma 7 to obtain $B(K(\theta))^2 \ge C p^\beta_{\boldsymbol{\nu}}(\varepsilon)$, for some positive constant C. This proves the first statement in item 1. The proof of the second statement is identical, since $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| < 1/2$ implies that $\boldsymbol{\nu}_\ell \in \Delta_{j_1}(\varepsilon)$ for some j_1, so that if $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair then $\boldsymbol{\nu}' \notin \Delta_{j_1}(\varepsilon)$, and therefore $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}'| \ge C_3 p^\beta_{\boldsymbol{\nu}'}(\varepsilon)$.

To prove item 2, notice that $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \le C_1 C_2 p^{\alpha+\beta}_{\boldsymbol{\nu}}(\varepsilon)$ (cf. Remark 18). If $K(\theta) > B_2 p^{\beta/2}_{\boldsymbol{\nu}}(\varepsilon)$ then $K(\theta) \ge C |\boldsymbol{\nu} - \boldsymbol{\nu}'|^{\beta/2(\alpha+\beta)}$. If $K(\theta) \le B_2 p^{\beta/2}_{\boldsymbol{\nu}}(\varepsilon)$ then $\mathcal{P}(\ell_e,\ell_0)$ has only lines with $i_\ell = 0$, so that by item 3 in Lemma 7 one finds $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \le B K(\theta)^2$. ■

Lemma 11. *Given a tree $\theta \in \mathcal{S}_R$ such that $\widetilde{\mathfrak{D}}(\theta,\gamma) \cap \widetilde{\mathfrak{S}}(\theta,\gamma) \neq \emptyset$, if $N_h(\theta) \geq 1$ for some $h \geq 1$, then $cK(\theta)2^{(2-h)\beta/2\tau} \geq 1$, with c the same constant as in Lemma 8.*

Proof. Consider a tree $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,\bar{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \geq 1$, $\bar{h} \geq 1$, $\sigma,\sigma' = \pm$ and $\boldsymbol{\nu},\boldsymbol{\nu}' \in \mathfrak{O}$ such that $|\boldsymbol{\nu}'| \geq 2^{(\bar{h}-2)/\tau}$. Assume $N_h(\theta) \geq 1$ for some $\bar{h} \geq h \geq 1$.

If there is a line $\ell \in L(\theta)$, which does not belong to $\mathcal{P} := \mathcal{P}(\ell_e,\ell_0)$, such that $h_\ell \geq h$, then one can reason as at the beginning of the proof of Lemma 8 to obtain $K(\theta)E_h^{-1} \geq 2$, with $E_h = c^{-1}2^{(h-2)\beta/2\tau} \geq 1$.

Otherwise, there are lines $\ell \in \mathcal{P}$ on scale $h_\ell \geq h$, and hence such that $i_\ell = 1$ and, consequently, $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair. Let $\bar{\ell}$ be the one closest to ℓ_e among such lines; thus, one has $|\boldsymbol{\nu}'_{\bar{\ell}} - \boldsymbol{\nu}'| \geq C_3 p^\beta_{\boldsymbol{\nu}'}(\varepsilon)$, so that one obtains $B(K(\theta))^2 \geq C p^\beta_{\boldsymbol{\nu}'}(\varepsilon) \geq C 2^{(\bar{h}-2)\beta/\tau}$, for some positive constant C. So, the desired bound follows once more. ■

Lemma 12. *Given a tree $\theta \in \mathcal{S}_R$ such that $\widetilde{\mathfrak{D}}(\theta,\gamma) \cap \widetilde{\mathfrak{S}}(\theta,\gamma) \neq \emptyset$, for all $h \geq 1$ one has*

$$N_h(\theta) \leq c\, K(\theta) 2^{(2-h)\beta/2\tau},$$

where c is the same constant as in Lemma 8.

Proof. Consider a tree $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,\bar{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \geq 1$, $\bar{h} \geq 1$, $\sigma,\sigma' = \pm$ and $\boldsymbol{\nu},\boldsymbol{\nu}' \in \mathfrak{O}$ such that $|\boldsymbol{\nu}'| \geq 2^{(\bar{h}-2)/\tau}$.

For $k(\theta) = 1$ one has $N_h(\theta) \leq 1$, so that the bound follows from Lemma 11.

For $k(\theta) > 1$ one can proceed as follows. Let ℓ_0 be the root line of θ and call $\theta_1,\ldots,\theta_m$ the subtrees of θ whose root lines $\ell_1,\ldots,\ell_m$ are the lines on scale $h_{\ell_i} \geq h-1$ and $i_{\ell_i} = 1$ which are the closest to ℓ_0. All the trees θ_i such that $\ell_i \notin \mathcal{P}(\ell_e,\ell_0)$ belong to some $\Theta^{(k_i)\pm}_{R,\boldsymbol{\nu}_i}$ with $k_i < k$. If $K(\theta) \geq B_2 p^{\beta/2}_{\boldsymbol{\nu}'}(\varepsilon)$ (cf. Lemma 10) it may be possible that a line, say ℓ_1, belongs to $\mathcal{P}(\ell_e,\ell_0)$, so that $\mathrm{Val}(\theta_1) = g_{\ell_1}\,\mathrm{Val}(\theta'_1)$, with $\theta'_1 \in \mathcal{S}^{(k_1),\sigma_1,\sigma'}_{R,h_1,\boldsymbol{\nu}_1,\boldsymbol{\nu}'}$ with $h_1 \leq \bar{h}$, $\sigma_1 = \pm$ and $k_1 < k$.

If $h_{\ell_0} < h$ one has $N_h(\theta) = N_h(\theta_1) + \ldots + N_h(\theta_m)$, so that the bound $N_h(\theta) \leq K(\theta)E_h^{-1}$ follows by the inductive hypothesis.

If $h_{\ell_0} \geq h$ one has $N_h(\theta) = 1 + N_h(\theta_1) + \ldots + N_h(\theta_m)$. For $m = 0$ the bound can be obtained once more from Lemma 11, while for $m \geq 2$ at least one tree, say θ_m, belongs to $\Theta^{(k')\pm}_{R,\boldsymbol{\nu}'}$ for some k' and $\boldsymbol{\nu}'$ so that we can apply Lemma 8 and the inductive hypothesis to obtain

$$\begin{aligned} N_h(\theta) &\leq 1 + \left(K(\theta_1) + \ldots + K(\theta_{m-1})\right) E_h^{-1} + \big(K(\theta_m)E_h^{-1} - 1\big) \\ &\leq \left(K(\theta_1) + \ldots + K(\theta_{m-1})\right) E_h^{-1} + K(\theta_m)E_h^{-1} \leq K(\theta)E_h^{-1}, \end{aligned}$$

which yields the bound.

Finally if $m = 1$ one has $N_h(\theta) = 1 + N_h(\theta_1)$. Hence, if $\ell_1 \notin \mathcal{P}(\ell_e,\ell_0)$, again the bound follows from Lemma 8. If on the contrary $\ell_1 \in \mathcal{P}(\ell_e,\ell_0)$, one can adapt the discussion of the case $m = 1$ in the proof of Lemma 8. ■

Lemma 13. *There exists positive constants κ, ξ_1 and D_1 such that, if $\xi > \xi_1$ in Definition 8, then for*

all trees $\theta \in \mathcal{R}_R$ and for all $(\varepsilon, M) \in \widetilde{\mathfrak{D}}(\theta,\gamma) \cap \widetilde{\mathfrak{S}}(\theta,\gamma)$, by setting $\boldsymbol{\nu} = \boldsymbol{\nu}'_{\ell_0}$ and $\boldsymbol{\nu}' = \boldsymbol{\nu}_{\ell_e}$, one has

$$|\mathrm{Val}(\theta)| \le D_1^k 2^{-h} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{5.6a}$$

$$|\partial_\varepsilon \mathrm{Val}(\theta)| \le D_1^k 2^{-h} p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)\, \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{5.6b}$$

$$\sum_{\boldsymbol{\nu}_1\in\mathfrak{O}} \sum_{\boldsymbol{\nu}_2\in\mathcal{C}_{\boldsymbol{\nu}_1}} \sum_{\sigma_1,\sigma_2=\pm} \left|\partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} \mathrm{Val}(\theta)\right| \le D_1^k 2^{-h} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{5.6c}$$

with ρ as in Lemma 10.

Proof. Set for simplicity $\mathcal{P} = \mathcal{P}(\ell_e, \ell_0)$ and

$$\begin{aligned}
\Sigma(\theta) &= \sum_{v\in V_0(\theta)} |m_v| + \sum_{\ell\in L_q(\theta)} |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell| + \sum_{v\in E(\theta)} |\boldsymbol{\nu}_v|,\\
\Pi(\theta) &= \Big(\prod_{v\in V_0(\theta)} \mathrm{e}^{A_2|m_v|/8}\Big)\Big(\prod_{\ell\in L_q(\theta)} \mathrm{e}^{\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}\Big)\Big(\prod_{v\in E(\theta)} \mathrm{e}^{\lambda_0|\boldsymbol{\nu}_v|}\Big).
\end{aligned}$$

If $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \ge 1$, $h \ge 1$, $\sigma,\sigma' = \pm$ and $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$ resonant, then $N_h(\theta) \ge 1$, so that $K(\theta) = k + \Sigma(\theta) > C2^{h\beta/2\tau}$, for some constant C, which imply $1 \le 2^{-h} C^k \Pi(\theta)$, for some constant C. This produces the extra factor 2^{-h}.

By item 2 in Lemma 10 one has $(B_3^{-1}|\boldsymbol{\nu}-\boldsymbol{\nu}'|)^\rho \le K(\theta)$, so that $1 \le \mathrm{e}^{-|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} C^k\Pi(\theta)$, for some constant C. The factor $\Pi(\theta)$ can be bounded by using part of the factors $\mathrm{e}^{-A_2|m_v|}$, $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|}$, and $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}$, associated with the nodes and with the q-lines. This proves the bound (5.6a),

To prove the bound (5.6b) one has to take into account the further ε-derivative acting on the line propagator g_ℓ, for some $\ell \in L(\theta)$. If the line ℓ does not belong to $\mathcal{P}$ then one can reason as in the proof of (5.4b) in Lemma 9. If $\ell \in \mathcal{P}$ one has to distinguish between two cases. If there exists a line $\bar\ell \in \mathcal{P}$ such that $i_{\bar\ell} = 1$ then $K(\theta) > B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$ by item 1 in Lemma 10, so that, by item 2 in Lemma 7, one has $p_{\boldsymbol{\nu}_\ell}(\varepsilon) \le |\boldsymbol{\nu}'_\ell| \le B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha} \le B(2p_{\boldsymbol{\nu}}(\varepsilon) + K(\theta))^{1+4\alpha} \le C(K(\theta))^{4/\beta}$, for some constant C. If $i_\ell = 0$ for all lines $\ell \in \mathcal{P}$ then, by item 3 in Lemma 7, one has $p_{\boldsymbol{\nu}_\ell}(\varepsilon) \le |\boldsymbol{\nu}'_\ell| \le |\boldsymbol{\nu}_{\ell_e}| + B(K(\theta))^2$. Then item 3 in Hypothesis 1 implies the bound (5.6b).

To prove (5.6c) one has to study a sum of terms each containing a derivative $\partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} g_\ell$, for some $\ell \in L(\theta)$. If $\ell \in \mathcal{P}$ we distinguish between the two cases. If $K(\theta) > B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$, the sum over $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2$ has the limitations $|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2| \le C p_{\boldsymbol{\nu}_1}^{\alpha+\beta}(\varepsilon)$, $|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_\ell| \le C p_{\boldsymbol{\nu}_1}^{\alpha+\beta}(\varepsilon)$ and $|\boldsymbol{\nu}_\ell| \le (|\boldsymbol{\nu}_{\ell_e}| + BK(\theta))^{1+4\alpha} \le C(K(\theta))^{4/\beta}$, for some constant C: hence the sum over $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2$ produces a factor $C(K(\theta))^{C'}$ for suitable constants C and C', and one has $(K(\theta))^{C'} \le C^k\Pi(\theta)$, for some constant C. If $K(\theta) \le B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$, then $i_\ell = 0$ for all lines $\ell \in \mathcal{P}$, so that the line propagators g_ℓ do not depend on M. Finally if $\ell \notin \mathcal{P}$ then one has $|\boldsymbol{\nu}_\ell| \le B(K(\theta))^{1+4\alpha}$, so that the sum over $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2$ is bounded once more proportionally to $(K(\theta))^{C'}$, for some constant C', and again one can bound $(K(\theta))^{C'} \le C^k\Pi(\theta)$, for some constant C. ∎

Remark 20. *Both Lemma 12 and 16 deal with the first derivatives of* Val(θ)*. One can easily extend the analysis so to include derivatives of arbitrary order, at the price of allowing larger constants ξ_1 and D_1 – and a factor $p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)$ for any further ε-derivative. Therefore, one can prove that the function* Val(θ) *is C^r for any integer r, in particular for $r = 1$, which we shall need in the following – cf. in particular the forthcoming Lemma 14.*

6 Proof of Proposition 1

Definition 28. ***(The extended tree values).*** *Let the function χ_{-1} be as in Definition 11. Define*

$$\mathrm{Val}^E(\theta) = \Big(\prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} \chi_{-1}(|x_{\boldsymbol{\nu}_\ell}(\varepsilon)|\, p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon))\Big)\Big(\prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} \prod_{\boldsymbol{\nu}\in\mathcal{C}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar\gamma|\, |\boldsymbol{\nu}|^{\tau_1})\Big) \times \\ \times\Big(\prod_{\substack{\ell\in L_p(\theta)\\ i_\ell=0}} \chi_{-1}(||\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| - \bar\gamma|\, |\boldsymbol{\nu}_\ell|^{\tau_1})\Big)\mathrm{Val}(\theta) \tag{6.1}$$

for $\theta\in\Theta^{(k)}_{R,\boldsymbol{\nu}}$, and

$$\mathrm{Val}^E(\theta) = \Big(\prod_{\substack{\ell\in L(\theta)\setminus\{\ell_0,\ell_e\}\\ i_\ell=1}} \chi_{-1}(|x_{\boldsymbol{\nu}_\ell}(\varepsilon)|\, p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon))\Big)\Big(\prod_{\substack{\ell\in L(\theta)\setminus\{\ell_0,\ell_e\}\\ i_\ell=1}} \prod_{\boldsymbol{\nu}\in\mathcal{C}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar\gamma|\, p^{\tau_1}_{\boldsymbol{\nu}_\ell}(\varepsilon))\Big) \times \\ \times\Big(\prod_{\substack{\ell\in L_p(\theta)\\ i_\ell=0,\ \boldsymbol{\nu}_\ell\notin\mathcal{C}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| - \bar\gamma|\, |\boldsymbol{\nu}_\ell|^{\tau_1})\Big)\mathrm{Val}(\theta) \tag{6.2}$$

for $\theta\in\mathcal{R}^{(k)}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$. We call $\mathrm{Val}^E(\theta)$ the extended value *of the tree θ.*

The following result proves Proposition 1.

Lemma 14. *Given $\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, the function $\mathrm{Val}(\theta)$ can be extended to the function (6.1) defined and C^1 in $\mathfrak{D}_0\setminus\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, such that, defining the "extended" counterterm $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ according to Definition 25, with $\mathrm{Val}(\theta)$ replaced with $\mathrm{Val}^E(\theta)$, the following holds.*

1. *Possibly with different constants ξ_1 and K_0, $\mathrm{Val}^E(\theta)$ satisfies for all $(\varepsilon,M)\in\mathfrak{D}_0\setminus\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ the same bounds in Lemma 13 as $\mathrm{Val}(\theta)$ in $\mathfrak{D}(\gamma)$.*
2. *There exist constants ξ_1, K_1, κ, ρ and η_0, such that, if $\xi>\xi_1$ in Definition 8, $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ satisfies, for all $(\varepsilon,M)\in\mathfrak{D}_0\setminus\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ and $|\eta|\le\eta_0$, the bounds*

$$\begin{aligned} &\Big|L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\Big| \le |\eta|^N K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho}, \qquad \Big|\partial_\varepsilon L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\Big| \le |\eta|^N K_1 p^{c_0}_{\boldsymbol{\nu}} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho},\\ &\Big|\partial_\eta L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\Big| \le N\,|\eta|^{N-1} K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho},\\ &\sum_{\boldsymbol{\nu}_1\in\mathfrak{O}\,,\sigma_1=\pm}\ \sum_{\boldsymbol{\nu}_2\in\mathcal{C}_{\boldsymbol{\nu}_1}\,,\sigma_2=\pm} \Big|\partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\Big| \mathrm{e}^{\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \le |\eta|^N K_1. \end{aligned}$$

3. *$\mathrm{Val}^E(\theta)=\mathrm{Val}(\theta)$ for $(\varepsilon,M)\in\mathfrak{D}(2\gamma)$ and $\mathrm{Val}^E(\theta)=0$ for $(\varepsilon,M)\in\mathfrak{D}_0\setminus\mathfrak{D}(\gamma)$.*

Analogously, given $\theta\in\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$, the function $\mathrm{Val}(\theta)$ can be extended to the function (6.2) defined and C^1 in $\mathfrak{D}_0$, such that, defining $u^{E\,(k)}_{\boldsymbol{\nu}}$ as in Lemma 6 with $\mathrm{Val}(\theta)$ replaced with $\mathrm{Val}^E(\theta)$, the following holds.

1. *Possibly with different constants ξ_1 and K_0, $\mathrm{Val}^E(\theta)$ satisfies for all $(\varepsilon,M)\in\mathfrak{D}_0$ the same bounds in Lemma 9 as $\mathrm{Val}(\theta)$ in $\mathfrak{D}(\gamma)$.*
2. *There exist constants ξ_1, K_1, κ and η_0 such that, if $\xi>\xi_1$ in Definition 8, $u^{E\,\sigma}_{\boldsymbol{\nu}}$ satisfies, for all $(\varepsilon,M)\in\mathfrak{D}_0$ and $|\eta|\le\eta_0$, the bounds*

$$\big|u^{E\,\sigma}_{\boldsymbol{\nu}}\big| \le |\eta|^N K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}|^{1/2}}$$

   for all $\boldsymbol{\nu}\in\mathbb{Z}^{D+1}$.

3. $\mathrm{Val}^E(\theta) = \mathrm{Val}(\theta)$ *for* $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$ *and* $\mathrm{Val}^E(\theta) = 0$ *for* $(\varepsilon, M) \in \mathfrak{D}_0 \setminus \mathfrak{D}(\gamma)$.

Proof. We shall consider explicitly the case of trees $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$. The case of trees $\theta \in \Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ can be discussed in the same way.

Item 3 follows from the very definition. The bounds of item 1 can be proved by reasoning as in Section 5, by taking into account the further derivatives which arise because of the compact support functions χ_{-1} in (6.2). On the other hand all such derivatives produce factors proportional to $p^{a_2}_{\boldsymbol{\nu}_\ell}(\varepsilon)$ for some constant a_2 (again we refer to [18] for details); in particular we are using item 2 in Hypothesis 1 to bound the derivatives of $\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)$ with respect to ε. Therefore by using Lemma 8 and possibly taking larger constants ξ_1 and K_0 the bounds of Lemma 13 follow also for the extended function (6.2).

Finally the bounds on L^E in item 2 come directly from the definition. Indeed, the counterterms $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ are expressed in terms of the values $\mathrm{Val}(\theta)$ according to Remark 15, and the factor 2^{-h} is used to perform the summation over the scale labels. Hence we have to control the sum over the trees.

Let us fix ε. For each $v \in E(\theta)$ the sum over $|\boldsymbol{\nu}_v|$ is controlled by using the exponential factors $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|}$. For each line $\ell \in L(\theta)$ the labels $\boldsymbol{\nu}'_\ell$ are fixed by the conservation rule of item 12 in Definition 15, while the sum over $\boldsymbol{\nu}_\ell$ gives a factor $C_1 p^{\alpha}_{\boldsymbol{\nu}_\ell}(\varepsilon)$ for the p-lines (see item 2 in Hypothesis 3), and it is controlled by using the exponential factors $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}$ for the q-lines. The sums over i_ℓ and h_ℓ can be bounded by a factor 4. Finally the sum over all the unlabelled trees of order k is bounded by C^k for some constant C. Thus, the bounds on $L^E_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ are proved.

Finally, the C^1 smoothness follows from Remark 20. ∎

7 Proof of Proposition 2

The following result proves item 1 in Proposition 2. Here and henceforth we write $L = L(\eta, \varepsilon, M)$ and $L^E = L^E(\eta, \varepsilon, M)$, and we fix $\eta = \varepsilon^{1/N}$.

Lemma 15. *There exists constants* $\varepsilon_0 > 0$ *such that there exist functions* $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = M^{\sigma,\sigma'}_{\boldsymbol{\nu}',\boldsymbol{\nu}}(\varepsilon)$ *well defined and* C^1 *for* $\varepsilon \in \mathfrak{E}_0 \setminus \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, *such that the "extended" compatibility equation*

$$M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$$

holds for all $\varepsilon \in (0, \varepsilon_0) \setminus \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$.

Proof. By definition we set $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = 0$ for all ε such that $\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon)) = 0$. Consider the Banach space $\overline{\mathcal{B}}$ of lists $\{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\}$, with $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$ a resonant pair, such that each $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)$ is well defined and C^1 in $\varepsilon \in \mathfrak{E}_0 \setminus \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ and $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = 0$ for $\varepsilon \in \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$. By definition $\{L^{E\,\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}(\varepsilon^{1/N}, \varepsilon, \{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\})\}$ is well defined as a continuously differentiable application from $\overline{\mathcal{B}}$ in itself, since, for each tree $\theta \in \mathcal{R}^{(k)\sigma_1,\sigma_2}_{R,h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}$, the value $\mathrm{Val}^E(\theta)$ by definition smoothes out to zero the value of each line propagator g_ℓ in the corresponding intervals $\overline{\mathcal{I}}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}(2\gamma) \setminus \overline{\mathcal{I}}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}(\gamma)$. Again by definition $L^E(0,0,0) = 0$ and $|\partial_M L(0,0,0)|_{\mathrm{op}} = 0$, so that we can apply the implicit function theorem. ∎

Now we pass to the proof of item 2 in Proposition 2. We need some preliminary results.

Lemma 16. *Let* $A = A(\varepsilon)$ *a self-adjoint matrix piecewise differentiable in the parameter* ε. *Then, if* $\lambda^{(a)}(A)$ *and* $\phi^{(a)}(A)$ *denote the eigenvalues and the (normalised) eigenvectors of* A, *respectively, the following holds.*

1. *One has* $|\lambda^{(a)}(A(\varepsilon))| \le \|A(\varepsilon)\|_2$.

2. The eigenvalues $\lambda^{(a)}(A(\varepsilon))$ are piecewise differentiable in ε.

3. One has $|\partial_\varepsilon \lambda^{(a)}(A(\varepsilon))| \le \|\partial_\varepsilon A(\varepsilon)\|_2$.

Proof. See [19] for items 1 and 2. Moreover, for each interval in which A is differentiable, let A_n be an analytic approximation of A in such an interval, with $A_n \to A$ as $n \to \infty$: then the eigenvalues $\phi^{(a)}(A_n)$ are piecewise differentiable [19], and one has

$$\partial_\varepsilon \lambda^{(a)}(A_n) = \partial_\varepsilon \left(\phi^{(a)}, A_n \phi^{(a)}\right) = \lambda^{(a)}(A_n)\partial_\varepsilon \left(\phi^{(a)}, \phi^{(a)}\right) + \left(\phi^{(a)}, \partial_\varepsilon A_n \phi^{(a)}\right) = \left(\phi^{(a)}, \partial_\varepsilon A_n \phi^{(a)}\right),$$

which yields item 3 when the limit $n \to \infty$ is taken. ∎

For $M \in \mathcal{B}_\kappa$ we can write $\widehat{M} = \bigoplus_j M_j$, where M_j are block matrices, so that we can define $\|\widehat{M}\|_2 = \sup_j \|M_j\|_2$, with $\|M_j\|_2$ given as in Definition 7.

Lemma 17. *For $M \in \mathcal{B}_\kappa$ one has $\|\widehat{M}\|_2 \le C\varepsilon_0$ for some constant C depending on κ and ρ.*

Proof. If $M \in \mathcal{B}_\kappa$ then $\widehat{M} = \bigoplus_j M_j$, with M_j a block matrix with dimension d_j depending on j, and $M_j(a,b) = M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$, for suitable $\boldsymbol{\nu}, \boldsymbol{\nu}', \sigma, \sigma'$ such that $|M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}| \le D\varepsilon_0 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho}$ for some constant D. Therefore

$$\begin{aligned}
\|M_j\|_2^2 &= \max_{|x|_2 \le 1} |M_j x|_2^2 \le \max_{|x|_2 \le 1} \sum_{a,b,c=1}^{d_j} |M_j(a,b)|\,|x(b)|\,|M_j(a,c)|\,|x(c)| \\
&\le \frac{1}{2} \max_{|x|_2 \le 1} \sum_{a,b,c=1}^{d_j} |M_j(a,b)|\,|M_j(a,c)| \left(|x(b)|^2 + |x(c)|^2\right) \\
&\le \max_{|x|_2 \le 1} \sum_{a=1}^{d_j} |M_j(a,b)| \sum_{c=1}^{d_j} |M_j(a,c)| \sum_{b=1}^{d_j} |x(b)|^2 \le \left(\sum_{a=1}^{d_j} |M_j(a,b)| \right)^2,
\end{aligned}$$

which yields the assertion. ∎

Lemma 18. *Let A, B be two self-adjoint $d \times d$ matrices. Then*

$$\left|\lambda^{(a)}(A+B) - \lambda^{(a)}(A)\right| \le \sum_{b=1}^{d} \left|\lambda^{(b)}(B)\right|$$

for all $a = 1, \ldots, d$.

Proof. The result follows from Lidskii's lemma; cf. [19]. ∎

Define $\mathfrak{E}_1 = \{\varepsilon \in \mathfrak{E}_0 : x_{\boldsymbol{\nu}}(\varepsilon) \ge 2\gamma/p^\tau_{\boldsymbol{\nu}}(\varepsilon)\ \forall \boldsymbol{\nu} \in \mathfrak{O}\}$ and $\mathfrak{E}_2 = \{\varepsilon \in \mathfrak{E}_0 : ||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}| \ge 2\gamma/|\boldsymbol{\nu}|^{\tau_1}\ \forall \boldsymbol{\nu} \in \mathfrak{O}\}$, and set $\mathfrak{E} = \mathfrak{E}_1 \cap \mathfrak{E}_2$.

We can denote by $\lambda^\sigma_{\boldsymbol{\nu}}(A)$, with $\boldsymbol{\nu} \in \mathfrak{O}$ and $\sigma = \pm$, the eigenvalues of the block matrix $A = \mathbb{D} + \widehat{M}$. If $|\delta_{\boldsymbol{\nu}}(\varepsilon)| \ge \bar{\gamma}$ then $\lambda^\sigma_{\boldsymbol{\nu}}(\varepsilon) = \delta_{\boldsymbol{\nu}}(\varepsilon)$. Moreover for each $\varepsilon \in \mathfrak{E}_0$ and each $\boldsymbol{\nu} \in \mathfrak{O}$ such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$, there exists a block $A^{\boldsymbol{\nu}}(\varepsilon)$ of the matrix A, of size $d_{\boldsymbol{\nu}}(\varepsilon) \le 2C_1 p^\alpha_{\boldsymbol{\nu}}(\varepsilon)$ such that $\lambda^\pm_{\boldsymbol{\nu}}(A)$ depends only on the entries of such a block. This follows from Remarks 6 and 12.

Therefore we have to discard from $\mathfrak{E}_0$ only values of ε such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ for some $\boldsymbol{\nu} \in \mathfrak{O}$: for all such $\boldsymbol{\nu}$ the matrix $A^{\boldsymbol{\nu}}(\varepsilon)$ is well defined, and one has $\lambda^\sigma_{\boldsymbol{\nu}}(A) = \lambda^\sigma_{\boldsymbol{\nu}}(A_{\boldsymbol{\nu}}(\varepsilon))$.

One has, by item 3 in Lemma 4,

$$x_{\boldsymbol{\nu}}(\varepsilon) \ge \frac{1}{p^\xi_{\boldsymbol{\nu}}(\varepsilon)} \min_{a=1,\ldots,d_{\boldsymbol{\nu}}(\varepsilon)} \left|\lambda^{(a)}(A^{\boldsymbol{\nu}}(\varepsilon))\right| \ge \frac{1}{p^\xi_{\boldsymbol{\nu}}(\varepsilon)} \min_{\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} \min_{\sigma=\pm} |\lambda^\sigma_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}}(\varepsilon))|, \tag{7.1}$$

so that, by using that $\lambda^{\sigma}_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}}(\varepsilon)) = \lambda^{\sigma}_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}'}(\varepsilon)) = \lambda^{\sigma}_{\boldsymbol{\nu}'}(A)$ for all $\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$, we shall impose the conditions

$$|\lambda^{\sigma}_{\boldsymbol{\nu}}(A^{\boldsymbol{\nu}}(\varepsilon))| \geq \frac{\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}}, \qquad \boldsymbol{\nu} \in \mathfrak{O}, \qquad \sigma = \pm, \tag{7.2}$$

for suitable $\gamma_2 > 2\gamma$. Thus, the conditions (7.2), together with the bound $|\boldsymbol{\nu}| \leq 2p_{\boldsymbol{\nu}}(\varepsilon)$ (cf. Remark 18), will imply through (7.1) the bounds (5.3) for $x_{\boldsymbol{\nu}}(\varepsilon)$.

Define

$$\mathfrak{K}^{\sigma}_{\boldsymbol{\nu}} = \left\{ \varepsilon \in \mathfrak{E}_0 : |\lambda^{\sigma}_{\boldsymbol{\nu}}(A)| \leq \frac{\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}} \right\}, \qquad \boldsymbol{\nu} \in \mathfrak{O}, \qquad \sigma = \pm, \tag{7.3}$$

with $\tau_2 = \tau - \xi$, so that we can estimate

$$\operatorname{meas}(\mathfrak{E}_0 \setminus \mathfrak{E}_1) \leq \sum_{\boldsymbol{\nu} \in \mathfrak{O}} \sum_{\sigma = \pm} \operatorname{meas}(\mathfrak{K}^{\sigma}_{\boldsymbol{\nu}}). \tag{7.4}$$

Moreover, by defining

$$\mathfrak{H}_{\boldsymbol{\nu},\sigma} = \left\{ \varepsilon \in \mathfrak{E}_0 : |\delta_{\boldsymbol{\nu}}(\varepsilon) - \sigma\bar{\gamma}| \leq \frac{2\gamma}{|\boldsymbol{\nu}|^{\tau_1}} \right\}, \qquad \boldsymbol{\nu} \in \mathcal{C}_j, \quad j \in \mathbb{N}, \quad \sigma = \pm, \tag{7.5}$$

with τ_1 to be determined, one has

$$\operatorname{meas}(\mathfrak{E}_0 \setminus \mathfrak{E}_2) \leq \sum_{\boldsymbol{\nu} \in \mathbb{Z}^{D+1}} \sum_{\sigma = \pm} \operatorname{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}). \tag{7.6}$$

Lemma 19. *There exists constants w_0 and w_1 such that $\mathfrak{K}^{\pm}_{\boldsymbol{\nu}} = \emptyset$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \leq w_0/\varepsilon_0^{w_1}$. There exists constants y_0 and y_1 such that $\mathfrak{H}_{\boldsymbol{\nu},\pm} = \emptyset$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \leq y_0/\varepsilon_0^{y_1}$.*

Proof. We start by considering the sets $\mathfrak{K}^{\sigma}_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \in \mathfrak{O}$ and $\sigma = \pm$. If $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ one can write $A^{\boldsymbol{\nu}}(\varepsilon) = \operatorname{diag}\{\delta_{\boldsymbol{\nu}'}(0), \delta_{\boldsymbol{\nu}'}(0)\}_{\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} + B^{\boldsymbol{\nu}}(\varepsilon)$, which defines the matrix $B^{\boldsymbol{\nu}}(\varepsilon)$ as

$$B^{\boldsymbol{\nu}}(\varepsilon) = \operatorname{diag}\{\delta_{\boldsymbol{\nu}'}(\varepsilon) - \delta_{\boldsymbol{\nu}'}(0)\}^{\sigma=\pm}_{\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} + M^{\boldsymbol{\nu}}(\varepsilon),$$

where $M^{\boldsymbol{\nu}}(\varepsilon)$ is the block of $M(\varepsilon)$ with entries $M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}(\varepsilon)$ such that $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. By Lemma 18, one has

$$|\lambda^{\sigma}_{\boldsymbol{\nu}}(A) - \delta_{\boldsymbol{\nu}}(0)| \leq \sum_{a=1}^{d_{\boldsymbol{\nu}}(\varepsilon)} \left| \lambda^{(a)}(B^{\boldsymbol{\nu}}(\varepsilon)) \right| \leq 2C_1 p^{\alpha}_{\boldsymbol{\nu}}(\varepsilon) \|B^{\boldsymbol{\nu}}(\varepsilon)\|_2, \qquad \boldsymbol{\nu} \in \mathfrak{O}, \qquad \sigma = \pm, \tag{7.7}$$

where we have used Remark 10 to bound $d_{\boldsymbol{\nu}}(\varepsilon)$.

One has $|\delta_{\boldsymbol{\nu}}(0)| \geq \gamma_0/|\boldsymbol{\nu}|^{\tau_0} \geq \gamma_0/(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0}$ by item 2 in Hypothesis 1, whereas $\|B^{\boldsymbol{\nu}}(\varepsilon)\|_2 \leq c_2(2p_{\boldsymbol{\nu}}(\varepsilon))^{c_0}\varepsilon_0 + \|M^{\boldsymbol{\nu}}(\varepsilon)\|_2$, by items 1 and 2 in Hypothesis 1, and $\|M^{\boldsymbol{\nu}}(\varepsilon)\|_2 \leq \|M(\varepsilon)\|_2 \leq C_0\varepsilon_0$ by Lemma 17. Therefore (7.7) implies

$$|\lambda^{\sigma}_{\boldsymbol{\nu}}(A)| \geq \frac{\gamma_0}{(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0}} - Cp^{c_0+1}_{\boldsymbol{\nu}}(\varepsilon)\,\varepsilon_0,$$

for a suitable constant C, so that, by setting $w_1 = c_0 + 1 + \tau_0$ and choosing suitably the constants γ_2, τ and w_0, one has $|\lambda^{\pm}_{\boldsymbol{\nu}}(A)| \geq \gamma_0/2(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0} \geq \gamma_2/p^{\tau_2}_{\boldsymbol{\nu}}(\varepsilon)$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \leq w_0/\varepsilon_0^{w_1}$.

For the sets $\mathfrak{H}_{\boldsymbol{\nu},\sigma}$, one can reason in the same way, by using that $\bar{\gamma} \in \mathfrak{G}$ (cf. Definition 4). ∎

Lemma 20. *Let $\xi > \xi_1$ and $\varepsilon_0 = \eta_0^N$ be fixed as in Lemma 14. There exist constants γ, τ and τ_1 such that* $\operatorname{meas}(\mathfrak{E}_0 \setminus \mathfrak{E}) = o(\varepsilon_0)$.

Proof. First of all we have to discard from $\mathfrak{E}_0$ the sets $\mathfrak{H}_{\boldsymbol{\nu},\sigma}$. It is easy to see that one has

$$\mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le \frac{2\gamma}{|\boldsymbol{\nu}|^{\tau_1}} \frac{2}{c_1|\boldsymbol{\nu}|^{c_0}},$$

for some positive constant C, so that, by using the second assertion in Lemma 19, we find

$$\sum_{\boldsymbol{\nu}\in\mathfrak{O}} \sum_{\sigma=\pm 1} \mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le \sum_{\substack{\boldsymbol{\nu}\in\mathfrak{O} \\ |\boldsymbol{\nu}|\ge y_0/\varepsilon_0^{y_1}}} \sum_{\sigma=\pm 1} \mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le C\varepsilon_0^{y_1(\tau_1+c_0-D-1)},$$

for some constant C, provided $\tau_1+c_0-D>1$, so that we shall require for τ_1 to be such that $\tau_1+c_0-D>1$ and $y_1(\tau_1+c_0-D-1)>1$.

Next, we consider the sets $\mathfrak{K}^{\pm}_{\boldsymbol{\nu}}$. For all $\boldsymbol{\nu}\in\mathfrak{O}$ consider $A^{\boldsymbol{\nu}}(\varepsilon)$ and write $A^{\boldsymbol{\nu}}(\varepsilon)=\delta_{\boldsymbol{\nu}}(\varepsilon)I+B^{\boldsymbol{\nu}}(\varepsilon)$, which defines the matrix $B^{\boldsymbol{\nu}}(\varepsilon)$ as

$$B^{\boldsymbol{\nu}}(\varepsilon) = \mathrm{diag}\{\delta_{\boldsymbol{\nu}'}(\varepsilon)-\delta_{\boldsymbol{\nu}}(\varepsilon)\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} + M^{\boldsymbol{\nu}}(\varepsilon),$$

with $M^{\boldsymbol{\nu}}(\varepsilon)$ defined as in the proof of Lemma 19.

Then the eigenvalues of $A^{\boldsymbol{\nu}}(\varepsilon)$ are of the form $\lambda^{(a)}(A^{\boldsymbol{\nu}}(\varepsilon)) = \delta_{\boldsymbol{\nu}}(\varepsilon)+\lambda^{(a)}(B^{\boldsymbol{\nu}}(\varepsilon))$, so that for all $\varepsilon\in\mathfrak{E}_0\setminus\overline{\mathcal{I}}_{\boldsymbol{\nu}}(\gamma)$ one has

$$\left|\partial_\varepsilon \lambda^{(a)}(A^{\boldsymbol{\nu}})\right| \ge |\partial_\varepsilon\delta_{\boldsymbol{\nu}}(\varepsilon)| - \|\partial_\varepsilon B^{\boldsymbol{\nu}}(\varepsilon)\|_2\,,$$

where item 3 in Lemma 16 has been used. One has $|\partial_\varepsilon\delta_{\boldsymbol{\nu}}(\varepsilon)| \ge c_1|\boldsymbol{\nu}|^{c_0}$, by item 2 in Hypothesis 1, and $\|\partial_\varepsilon B^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le \max_{\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} |\partial_\varepsilon(\delta_{\boldsymbol{\nu}'}(\varepsilon)-\delta_{\boldsymbol{\nu}}(\varepsilon))| + \|\partial_\varepsilon M^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le \zeta c_3 p_{\boldsymbol{\nu}(\varepsilon)}\, p_{\boldsymbol{\nu}}^{c_0-1}(\varepsilon) + \varepsilon_0 C p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)$, for a suitable constant C, as follows from item 4 in Hypothesis 1, from Hypothesis 3 (see Remark 6 for the definition of ζ), from Lemma 14, and from Lemma 15. Hence we can bound $|\partial_\varepsilon\lambda^{(a)}(A^{\boldsymbol{\nu}})| \ge c_1|\boldsymbol{\nu}_0|^{c_0}/2$ for ε_0 small enough.

Therefore one has

$$\mathrm{meas}(\mathfrak{K}^{\sigma}_{\boldsymbol{\nu}}) \le \frac{2\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}(\varepsilon)} \frac{2}{c_1|\boldsymbol{\nu}|^{c_0}} \left(C|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)} \right), \tag{7.8}$$

for some constant $\overline{C}$, where the last factor $\overline{C}|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)}$ arises for the following reason. The eigenvalues $\lambda^{\sigma}_{\boldsymbol{\nu}}(A)$ are differentiable in ε except for those values ε such that for some $\boldsymbol{\nu}'\in\mathcal{C}_{\boldsymbol{\nu}}$ one has $|\delta_{\boldsymbol{\nu}'}(\varepsilon)|=\bar\gamma$ and $|\delta_{\boldsymbol{\nu}}(\varepsilon)|<\bar\gamma$. Because of item 3 in Hypothesis 1 all functions $\delta_{\boldsymbol{\nu}'}(\varepsilon)$ are monotone in ε as far as $|\delta_{\boldsymbol{\nu}'}(\varepsilon)|<1/2$, so that for each $\boldsymbol{\nu}'\in\mathcal{C}_{\boldsymbol{\nu}}$ the condition $|\delta_{\boldsymbol{\nu}'}(\varepsilon)|=\bar\gamma$ can occur at most twice. The number of $\boldsymbol{\nu}'\in\mathcal{C}_{\boldsymbol{\nu}}$ such that the conditions $|\delta_{\boldsymbol{\nu}'}(\varepsilon)|=\bar\gamma$ and $|\delta_{\boldsymbol{\nu}}(\varepsilon)|<\bar\gamma$ can occur for some $\varepsilon\in\mathfrak{E}_0$ is bounded by the volume of a sphere of centre $\boldsymbol{\nu}$ and radius proportional to $|\boldsymbol{\nu}|^{\alpha+\beta}$ (cf. Remark 6). Hence $\overline{C}|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)}$ counts the number of intervals in $\mathfrak{E}_0\setminus\overline{\mathcal{I}}_{\boldsymbol{\nu}}(\gamma)$.

Thus, (7.8) yields, by making use of the first assertion of Lemma 19,

$$\sum_{\boldsymbol{\nu}\in\mathfrak{O}} \sum_{\sigma=\pm} \mathrm{meas}(\mathfrak{K}^{\sigma}_{\boldsymbol{\nu}}) \le \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^{D+1} \\ |\boldsymbol{\nu}|\ge w_0/\varepsilon_0^{w_1}}} \frac{8\gamma}{c_1}|\boldsymbol{\nu}|^{-\tau_2-c_0}\left(\overline{C}|\boldsymbol{\nu}|^{2\alpha}\right) \le C\varepsilon_0^{w_1(\tau_2+c_0-2\alpha-D-1)},$$

for some positive constant C, provided $\tau_2+c_0-2\alpha-D=\tau+c_0-2\alpha-D-\xi>1$, so that (7.4) implies that $\mathrm{meas}(\mathfrak{E}_0\setminus\mathfrak{E}_1)\le C\varepsilon_0^{w_1(\tau_2+c_0-D-1)}$.

Therefore, the assertion follows provided $\min\{\tau_1,\tau_2-2\alpha\}>D-c_0+1$, $y_1(\tau_1+c_0-D-1)>1$ and $w_1(\tau_2+c_0-2\alpha-D-1)>1$. ∎

A Proof of Lemma 1

Lemma 1 is a consequence of the following elementary proposition in Galois theory.

Proposition 3. *If $p_1, \dots, p_k$ are distinct primes then the field*

$$F := \mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_k}]$$

obtained from the rational numbers $\mathbb{Q}$ by adding the k square roots $\sqrt{p_i}$ has dimension 2^k over $\mathbb{Q}$ with basis the elements $\prod_{i \in I} \sqrt{p_i}$ as I varies on the 2^k subsets of $\{1, 2, \dots, k\}$.

The group of automorphisms[2] *of F which fix $\mathbb{Q}$ (i.e. the Galois group of $F/\mathbb{Q}$) is an Abelian group generated by the automorphisms τ_i defined by $\tau_i(\sqrt{p_j}) = (-1)^{\delta(i,j)} \sqrt{p_j}$.*

Proof. We prove by induction both statements. Let us assume the statements valid for $p_1, \dots, p_{k-1}$ and let $F' := \mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_{k-1}}]$ so that $F = F'[\sqrt{p_k}]$. We first prove that $\sqrt{p_k} \notin F'$. Assume it to be false. Since $(\sqrt{p_k})^2$ is integer, each element – say τ – of the Galois group of $F'/\mathbb{Q}$ must either fix $\sqrt{p_k}$ or transform it into $-\sqrt{p_k}$ (by definition $\tau(p_k) = \tau(\sqrt{p}_k)^2 = p_k$).

Now any element $b \in F'$ is by induction uniquely expressed as

$$b = \sum_{I \subset \{1,2,\dots,k-1\}} a_I \prod_{i \in I} \sqrt{p_i}, \qquad a_I \in \mathbb{Q}.$$

If h of the numbers a_I are non zero, it is easily seen that b has 2^h transforms (changing the signs of each of the a_I) under the Galois group of F'. Therefore $b = \sqrt{p}_k$ if and only if $h = 1$, that is one should have $\sqrt{p_k} = m/n \prod_{i \in I} \sqrt{p_i}, I \subset \{1, 2, \dots, k-1\}$ for m, n integers. This implies that $p_k n^2 = m^2 \prod_{i \in I} p_i$ which is impossible by the unique factorisation of integers. This proves the first statement.

To construct the Galois group of $F/\mathbb{Q}$ we extend the action of τ_i for $i = 1, \dots, k-1$ by setting $\tau_i(\sqrt{p_k}) = \sqrt{p_k}$. Finally we define the automorphism τ_k as $\tau_k(\sqrt{p_j}) = (-1)^{\delta(k,j)} \sqrt{p_j}$ for $j = 1, \dots, k$. ■

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